Études in κ-$\mathbb{M}$-proper forcing.

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Abstract: κ-$\mathbb{M}$-proper forcing, introduced in [K00] when $\kappa = \omega_1$, is a very powerful new technique for generic stepping up, subsuming all previous generic steppings up using auxiliary functions. A general framework for using κ-$\mathbb{M}$-proper forcing is set out, and a couple of examples of such forcings, adding κ^--thin-very tall scattered spaces and long chains in $\mathcal{P}(\kappa)$ modulo $<\kappa^-$, are given. These objects are not currently obtainable by the previously known techniques.

Introduction.

As explained at some length in the introduction to [M*2], generic stepping up arguments are forcing constructions for building objects of size at least κ^+, for some successor cardinal κ, where one ensures that the forcing preserves cardinals by exploiting the cardinal preserving properties of a similar forcing to add a smaller object, usually of size κ, and the similarities between the two forcings. One also typically needs some sort of auxiliary combinatorial device to make transfers of properties between the forcings possible. Of course, such a stark definition does not provide much insight into the method, but after having seen a few examples one realises the power of the method and the appreciates the sublety of some of the auxiliary tools. Many of the most refined of these, for example, instantiate the failure of Chang's conjecture (and so assuming the consistency of mild large cardinals, ω_1-Erdős cardinals, themselves cannot be shown to exist in ZFC alone) and more.

Up until now the most incisive uses of these tools have mainly been limited to the situation in which $\kappa = \omega_1$. The most interesting exception perhaps occurs in [KM] where Koepke and Martinez show that the coupling function for a $(\kappa, 1)$-simplified morass for any successor cardinal κ has the 'new Δ-property,' a property introduced for $\kappa = \omega_1$ by Roitman in [R]. They then use the function as

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auxiliary apparatus to ensure the κ^+-chain condition holds for a stepped up forcing to generically add a certain sort of superatomic Boolean algebra of size κ^+.

As remarked in [M*2], this new Δ-property for arbitrary successor κ can be used to generalise to arbitrary successor κ other known forcing constructions that used a function with the new Δ-property for ω_1. Examples of this include the existence, given a $(\kappa^+, 1)$-simplified morass, of a κ^{++}-cc, $<\kappa^+$-closed forcing to ensure that $\binom{\kappa^{++}}{\kappa^{++}} \not\to \binom{\kappa}{\kappa}^{1,1}_{\kappa}$.

However proofs that *any* functions at all, including the coupling function and other well known functions such as Todorcevic's ρ-function, satisfy stronger properties such as the Knaster version of the new Δ-property or Veličković's Δ-property have so far proved elusive for arbitrary successor κ. This has hitherto frustrated efforts to extend known results on generically adding objects of size ω_2 to large double successor cardinals.

In this paper a couple of new examples of generic stepping up are given. The objects added are straightforward generalisations to arbitrary double successors of objects of size ω_2 added by well known generic stepping up proofs. In those proofs a single function witnessing the failure of Chang's conjecture and more is used to enable one to prove a requisite chain condition. However, lacking functions in the general case with the desirable properties that, for example, the coupling function enjoys when $\kappa = \omega_1$, the arguments in this paper use local connectedness functions from a $(\kappa, 1)$-simplified morass in order to show that the forcings are κ-$\mathbb{M}$-proper. This type of forcing, for $\kappa = \omega_1$, was first introduced by Piotr Koszmider in [K00] in order to resolve a problem that he showed to be unsusceptible to countable chain condition forcing. In [M*2] the technique was extended to arbitrary successor κ, and this paper relies on the general analysis of local distance functions and κ-$\mathbb{M}$-proper forcing given there.

The forcings add structures of size κ^+ for successor $\kappa \geq \omega_1$, using conditions which have working parts that are of size less than κ^-, the cardinal predecessor of κ, and using a set of maps of size less than κ^- from a $(\kappa, 1)$-simplified morass as side conditions. Put extremely briefly, the working parts of conditions in a generic for such forcing will make up collections of pieces that can be assembled into the structure(s) one is trying to add, while the side conditions provide constraints on the working parts that ensure that one can perform the amalgamations necessary to prove that κ is not collapsed to κ^-. Without these side condition constraints this cardinal collapsing would take place.

As discussed in the introduction to [M*2], κ-$\mathbb{M}$-proper forcing (with $\kappa = \omega_1$) is an outgrowth of the idea of using matrices of models as side conditions. So a feature of the technique which is novel in the context of generic stepping up is that instead of using auxiliary apparatuses (such the connectedness functions) to provide *a priori* constraints on all conditions, constraints are given *synthetically*. As such the arguments that cardinals are preserved are inevitably more delicate than arguments that forcings defined. The analysis given in [M*2, §4] of the relationship between the local connectedness functions of pairs of conditions which one needs to amalgamate in order to successfully complete a proof of κ-$\mathbb{M}$-properness plays a crucial rôle in these arguments.

Amongst other points to note from the analysis of [M*2, §4] is that a strengthening ([M*2, 4.24]) of the conclusion of the Veličković Δ-property holds for arbitrary successor κ — see the end of §2 below — for pairs of realms of conditions which one needs to amalgamate to prove cardinal preservation. This is the key to seeing that one can carry out using κ-$\mathbb{M}$-proper forcing any generic stepping up which one could do using a single function, with properties considered up to now and which are known to be consistent for $\kappa = \omega_1$, to show the κ^+-chain condition for a suitable forcing. Hence the claim in the abstract that κ-$\mathbb{M}$-proper forcing subsumes all previously considered generic steppings up using auxiliary functions.

It is hoped that besides the intrinsic interest of the examples, these *études* will prove useful illustrations of the new considerations in generic stepping up mentioned in the previous several paragraphs.

§1. Premliminaries about simplified morasses and local connected functions.

Most of the set theoretic notation used in the paper is standard, but it may be useful to remind the reader of a few items. If $X \subseteq \mathrm{On}$ then $\mathrm{ssup}(X)$, the *strong supremum of* X, is the least ordinal α such that $X \subseteq \alpha$, and if τ, ν are ordinals with $\tau < \nu$ then $[\tau, \nu)$ is the interval $\{\xi \in \mathrm{On} \mid \tau \leq \xi < \nu\}$. If χ, κ are cardinals with χ regular and $\chi < \kappa$ then $S^\kappa_\chi = \{\alpha < \kappa \mid \mathrm{cf}\,(\alpha) = \chi\}$. Of course, S^κ_χ is stationary in κ. If f is a function on increasing pairs of ordinals and ν, $\tau \in \mathrm{dom}(f)$ let $f\{\nu, \tau\}$ be $f(\nu, \tau)$ if $\nu < \tau$ and be $f(\nu, \tau)$ if $\nu < \tau$.

Definitions, Theorems, Lemmas and so on are numbered separately in each section. Thus, for example, Fact (3) of §1 is referred to as Fact (3) within §1 and as Fact (1.3) elsewhere.

I also recall relevant definitions and facts about simplified morasses.

Definition 1. ([V]) Let κ be a regular cardinal. $\mathbb{M} = \langle\langle\theta_\alpha \mid \alpha \leq \kappa\rangle, \langle\mathcal{F}_{\alpha\beta} \mid \alpha \leq \beta \leq \kappa\rangle\rangle$ *is a* $(\kappa, 1)$-*simplified morass* if $\langle\theta_\alpha \mid \alpha < \kappa\rangle$ is an increasing sequence of ordinals less than κ and $\theta_\kappa = \kappa^+$,

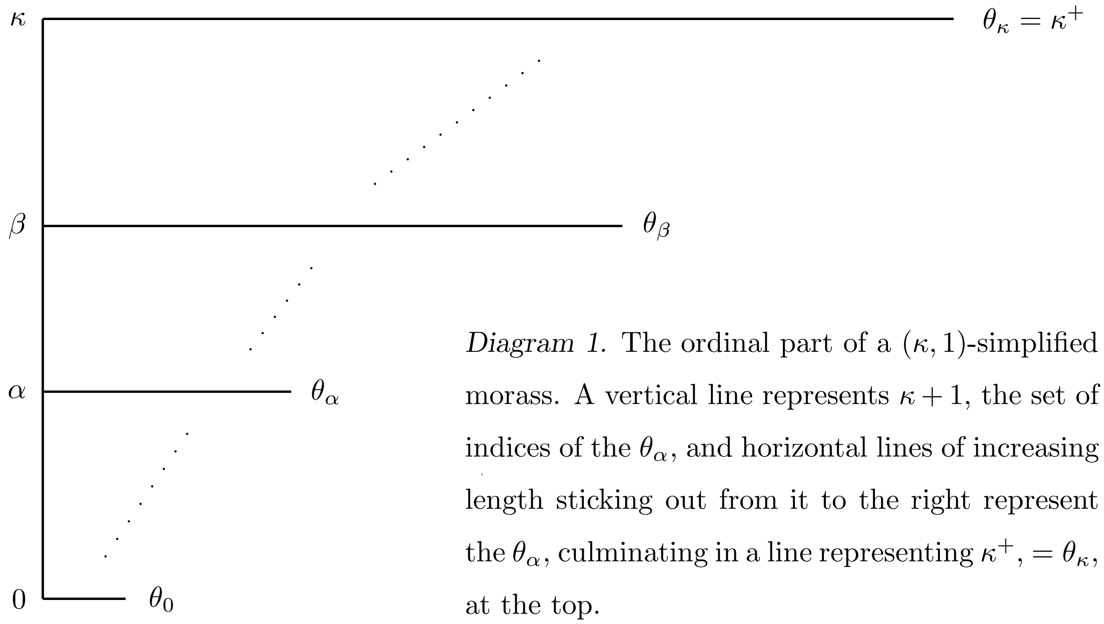

Diagram 1. The ordinal part of a $(\kappa, 1)$-simplified morass. A vertical line represents $\kappa + 1$, the set of indices of the θ_α, and horizontal lines of increasing length sticking out from it to the right represent the θ_α, culminating in a line representing κ^+, $= \theta_\kappa$, at the top.

and each $\mathcal{F}_{\alpha\beta}$ is a collection of maps from θ_α to θ_β

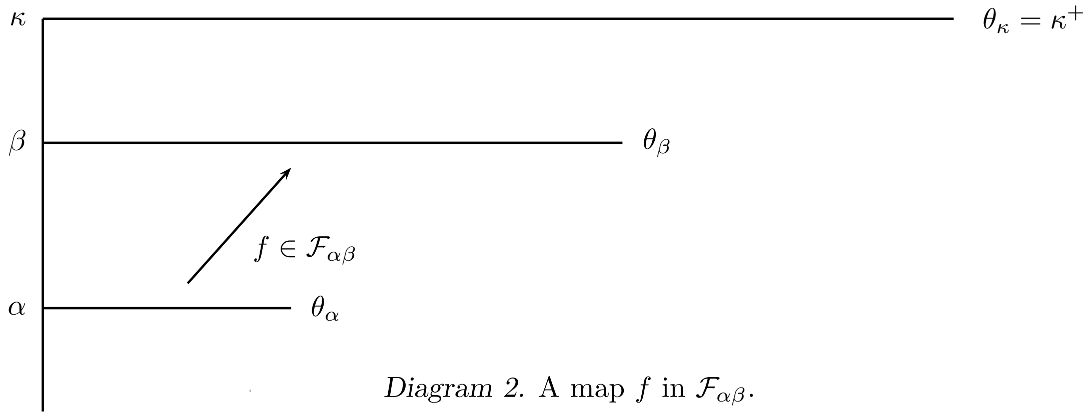

Diagram 2. A map f in $\mathcal{F}_{\alpha\beta}$.

such that the following properties hold:

(i) $\quad \forall\alpha \leq \kappa \;\; \mathcal{F}_{\alpha\alpha} = \{\mathrm{id}\}$

(ii) $\quad \forall\alpha \leq \beta \leq \gamma \leq \kappa \;\; \mathcal{F}_{\alpha\gamma} = \{\, g \cdot f \mid f \in \mathcal{F}_{\alpha\beta} \;\;\&\;\; g \in \mathcal{F}_{\beta\gamma}\,\}$

(iii) $\quad \forall\alpha < \kappa \; \big(\mathcal{F}_{\alpha\alpha+1} = \{\mathrm{id}\} \;\; \& \;\; \theta_{\alpha+1} = \theta_\alpha + 1 \;\text{ or }\; \mathcal{F}_{\alpha\alpha+1} = \{\mathrm{id},\, h\} \;\; \& \;\; \exists\sigma < \theta_\alpha \;\; \&$

$\qquad h \restriction \sigma = \mathrm{id} \;\; \& \;\; \forall\tau \; (\sigma + \tau < \theta_\alpha \longrightarrow h(\sigma + \tau) = \theta_\alpha + \tau)\big)$

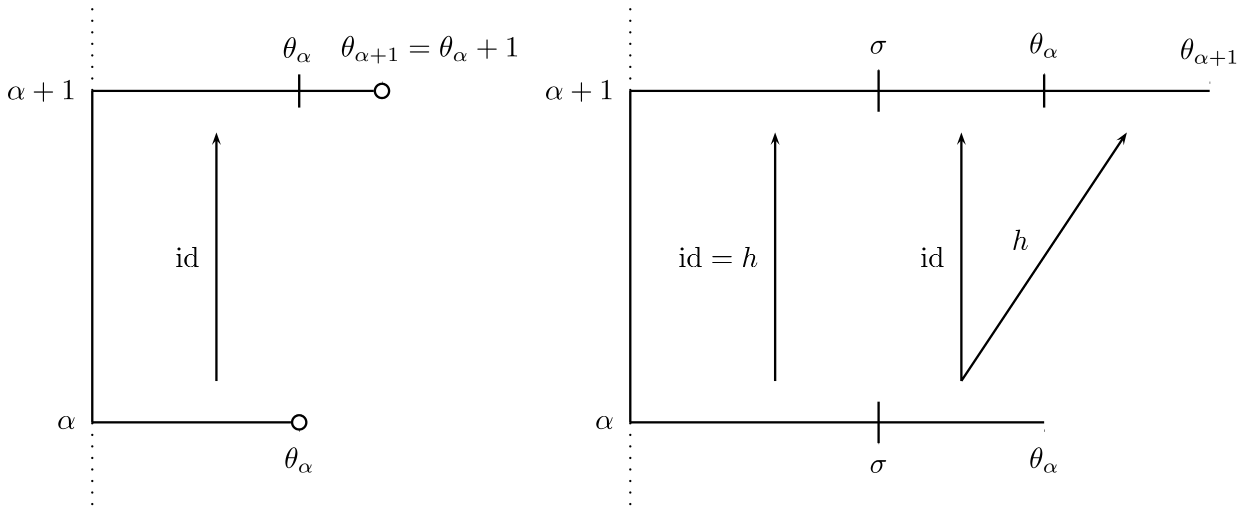

Diagram 3. Illustrating Definition (1.iii): for all $\alpha < \kappa$

$\mathcal{F}_{\alpha\alpha+1} = \{\mathrm{id}\}$ and $\theta_{\alpha+1} = \theta_\alpha + 1$ or $\mathcal{F}_{\alpha\alpha+1} = \{\mathrm{id}, h\}$, where $h \restriction \sigma = \mathrm{id} \restriction \sigma$ and $h(\sigma + \xi) = \theta_\alpha + \xi$ for all $\sigma_+\xi < \theta_\alpha$.

(iv) $\quad \forall\alpha \le \kappa \; \big(\alpha \text{ is a limit ordinal } \longrightarrow \forall\beta_0,\, \beta_1 < \alpha \; \forall f_0 \in \mathcal{F}_{\beta_0\alpha} \; \forall f_1 \in \mathcal{F}_{\beta_1\alpha}$

$\qquad \exists\gamma \in [\beta_0 \cup \beta_1, \alpha) \; \exists h \in \mathcal{F}_{\gamma\alpha} \; \exists g_0 \in \mathcal{F}_{\beta_0\gamma} \; \exists g_1 \in \mathcal{F}_{\beta_1\gamma} \; (f_0 = h \cdot g_0 \;\; \& \;\; f_1 = h \cdot g_1)\big)$

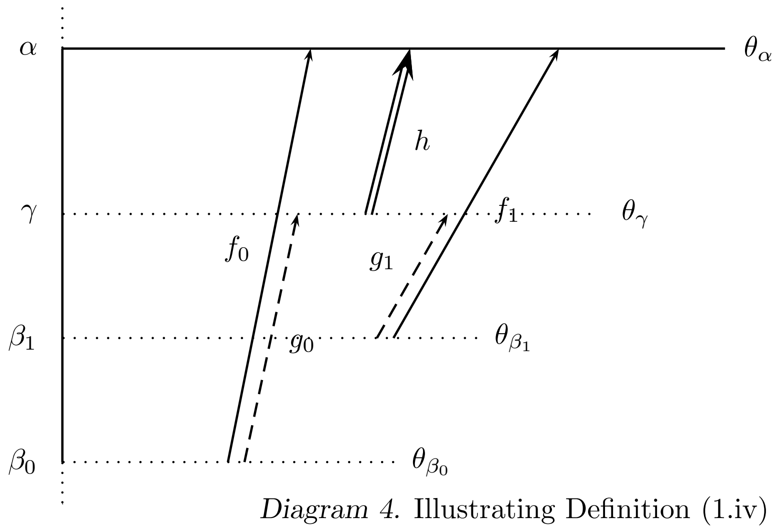

Diagram 4. Illustrating Definition (1.iv)

(v) $\quad \bigcup\{\, f``\theta_\alpha \mid \alpha < \kappa \;\; \& \;\; f \in \mathcal{F}_{\alpha\kappa} \,\} = \kappa^+.$

Definition 2. Let $\mathbb{M} = \langle\langle\theta_\alpha \mid \alpha \le \kappa\rangle, \langle\mathcal{F}_{\alpha\beta} \mid \alpha \le \beta \le \kappa\rangle\rangle$ be a $(\kappa, 1)$-simplified morass. Then $\mathcal{F} = \bigcup\{(\alpha, f) \mid \alpha < \kappa \ \& \ f \in \mathcal{F}_{\alpha\kappa}\}$.

(As a brief historical aside, Velleman's original definition of a $(\kappa, 1)$-simplified morass was with the properties of Definition (1) formulated in terms of the ranges of the maps in $\mathcal{F}$. But this definition, talking in terms of sets, elements and subsets, elements and subsets of those and so on, is terribly unweildy — one continually has to talk about and manipulate equivalence classes of sets under order isomorphism rather than working with ordinals for example. In fact, in his original paper [V], Velleman already made the transition to the formulation given above, in terms of ordinals and maps between them, and, because of its ease of use and understanding this has been the version of choice for virtually all authors since.)

The following well-known, fundamental fact due to Velleman always comes into play when dealing with simplified morasses.

Fact 3. (Velleman, [V, Lemma 3.2]) For any $\alpha \le \beta \le \kappa$, f, $g \in \mathcal{F}_{\alpha\beta}$ with $\nu \in \mathrm{rge}(f) \cap \mathrm{rge}(g)$ there is some $\overline{\nu} < \theta_\alpha$ such that $f(\overline{\nu}) = \nu = g(\overline{\nu})$ and $f \restriction \overline{\nu} + 1 = g \restriction \overline{\nu} + 1$.

Proof. By induction on β for each α.

Another two useful facts from [V] are the following.

Fact 4. (Stanley, [V, Theorem 3.9]) If $\alpha \le \beta \le \kappa$, $f \in \mathcal{F}_{\alpha\beta}$ and $\nu < \theta_\alpha$ then there is some $g \in \mathcal{F}_{\alpha\beta}$ such that $g \restriction \nu = f \restriction \nu$ and $g(\nu + \xi) = \mathrm{ssup}(g``\nu) + \xi$ for $\nu + \xi < \theta_\alpha$.

Proof. Again by induction on β for each α.

Fact 5. (Velleman, [V, Corollary 3.5]) If $\langle f_i \mid i < \chi\rangle$ is a collection of maps with each $f_i \in \mathcal{F}_{\alpha_i\beta}$ for $i < \chi$ and $\chi < \mathrm{cf}\,(\beta)$ then there is some $\alpha \in [\sup(\{\alpha_i \mid i < \chi\}, \beta)$, some $f \in \mathcal{F}_{\alpha\beta}$ and maps $g_i \in \mathcal{F}_{\alpha_i\alpha}$ for all $i < \chi$ such that $f_i = f \cdot g_i$ for all $i < \chi$. In particular any collection of fewer than κ maps in $\mathcal{F}$ can be factored through some single common map $(\alpha, f) \in \mathcal{F}$

The following notation is now more or less standard for simplified morasses (see, for example, [M]).

Definition 6. Let $\alpha \le \beta \le \kappa$, $f \in \mathcal{F}_{\alpha\beta}$ and $f(\overline{\nu}) = \nu$. Then $\psi_{\langle\alpha,\overline{\nu}\rangle,\langle\beta,\nu\rangle}$ is the function $f \restriction \overline{\nu} + 1$. By Fact (3) this is a good definition. If $\beta = \kappa$ write ψ^ν_α for $\psi_{\langle\alpha,\overline{\nu}\rangle,\langle\beta,\nu\rangle}$. Write ν_α for the $\overline{\nu}$ such that $\psi^{\nu}_{\alpha}(\overline{\nu}) = \nu$.

Definition 7. If $\beta \leq \kappa$ and $\nu < \tau < \theta_\beta$ then $c_\beta(\nu, \tau)$ = the least $\alpha \leq \beta$ such that there is some $f \in \mathcal{F}_{\alpha\beta}$ with ν, $\tau \in \mathrm{rge}(f)$. Write c for c_κ and note that $c(\nu, \tau) < \kappa$ for all ν, $\tau < \kappa^+$ by properties (iii) and (v) of gap-one simplified morasses. c is the *coupling function* for $\mathbb{M}$.

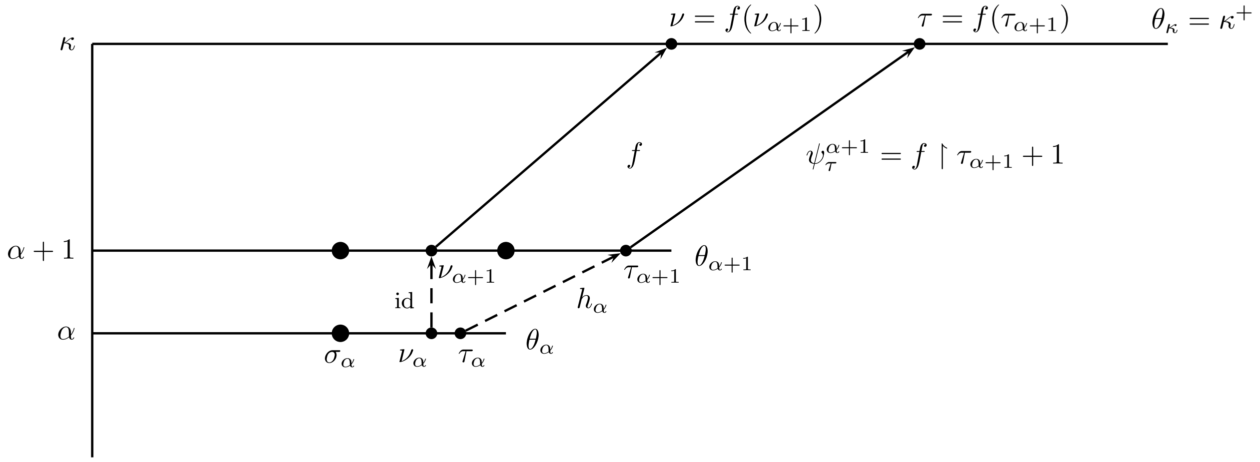

Diagram 5. Illustrating $c(\nu, \tau) = \alpha + 1$ being witnessed by $f \in \mathcal{F}_{\alpha+1\kappa}$. *cf.* Definitions (6) and (7).

As motivation for some of the properties of functions defined in §1 I also recall the following fact.

Fact 8. ([M]) c has the following properties:

If $\alpha < \kappa$ and $\tau < \kappa^+$ then $\overline{\overline{\{\xi \mid c(\xi, \tau) \leq \alpha\}}} < \kappa$.

c is *subadditive*: if $\mu < \nu < \tau < \kappa^+$ then

(i) $$c(\mu, \nu) \leq \max(\{\, c(\mu, \tau), c(\nu, \tau)\,\})$$

(ii) $$c(\mu, \tau) \leq \max(\{\, c(\mu, \nu), c(\nu, \tau)\,\})$$

If $\xi < \lambda = \bigcup \lambda < \mu$ there then is some $\zeta \in [\xi, \lambda)$ such that for all $\nu \in [\zeta, \lambda)$ one has $c(\nu, \lambda) \leq c(\nu, \mu)$.

Definition 9. Let $\nu < \tau < \kappa^+$ and suppose that $c_\kappa(\nu, \tau) \leq \alpha$. Then $D_\alpha(\nu, \tau) = \mathrm{otp}([\nu_\alpha, \tau_\alpha))$, *i.e.* $= \mathrm{otp}(\tau_\alpha \setminus \nu_\alpha)$. Let $D(\nu, \tau) = D_{c(\nu,\tau)}(\nu, \tau)$.

Definition 10. Let $\mathcal{L}$ be a first order language. If $\mathcal{A} = (A, \ldots)$ is an $\mathcal{L}$-structure then $|\mathcal{A}| = A$.

Definition 11. Let $\mathbb{M} = \langle \langle \theta_\alpha \mid \alpha \leq \kappa \rangle, \langle \mathcal{F}_{\alpha\beta} \mid \alpha \leq \beta \leq \kappa \rangle \rangle$ be a $(\kappa, 1)$-simplified morass. $\mathbb{M}$ is *stationary* if $\{\mathrm{rge}(f) \mid \exists \alpha\ (\alpha, f) \in \mathcal{F}\}$ is stationary in $[\kappa^+]^{<\kappa}$.

Now I turn to definitions concerned with the local connectedness functions discussed in [M*2].

Definition 12. ([M*2, Definition 1.2]) Let $A \subseteq \mathcal{F}$. Define $c_A(\nu,\tau) =$ the least $\alpha < \kappa$ such that there are $(\alpha_0, f_0), \dots (\alpha_{k-1}, f_{k-1}) \in A$, and $\xi_k = \nu \leq \xi_{k-1} \leq \dots \leq \xi_0 = \tau$ with

$$\forall i < k \ (\alpha_i \leq \alpha \ \ \& \ \ f_i \in \mathcal{F}_{\alpha_i \kappa} \quad \& \quad \xi_i, \xi_{i+1} \in \mathrm{rge}(f_i)),$$

if there is such an α and $= \kappa$ otherwise. $\langle \xi \,|\, i \leq k \rangle$ and $\langle (\alpha_i, f_i) \,|\, i < k \rangle$ are *witnesses* that $c(\nu,\tau) = \alpha$ if they are witnesses that $c(\nu,\tau) \leq \alpha$ and $\max(\{\alpha_i \,|\, i \leq k\}) = \alpha$. They are *good* witnesses if $\langle \alpha_i \,|\, i < k \rangle$ is strictly increasing.

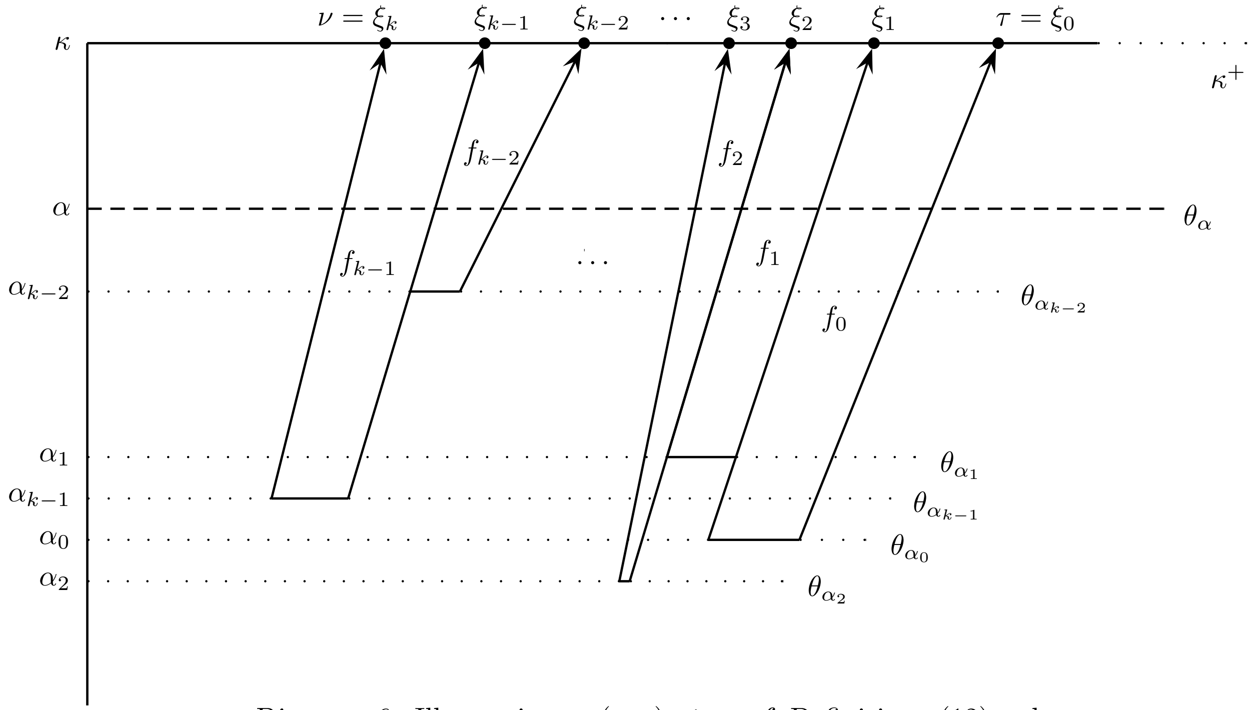

Diagram 6. Illustrating $c_A(\nu,\tau) \leq \alpha$, *cf.* Definitions (12), where $f_i \in \mathcal{F}_{\alpha_i\kappa}$, $(\alpha_i, f_i) \in A$ and ξ_i, $\xi_{i+1} \in \mathrm{rge}(f_i)$ for all $i < k$.

A couple of elementary facts are very useful.

Fact 13. ([M*2, Lemma 1.3]) $\forall A, B \subseteq \mathcal{F} \ \ \forall \nu < \tau < \kappa^+ \ \ (A \subseteq B \longrightarrow c_B(\nu,\tau) \leq c_A(\nu,\tau))$

Fact 14. ([M*2, Lemma 1.4]) Suppose that $A \subseteq \mathcal{F}$ and $\nu < \tau < \kappa^+$. If $f \in \mathcal{F}_{\beta\kappa}$, $\tau \in \mathrm{rge}(f)$ and $c_A(\nu,\tau) \leq \beta$ then $\nu \in \mathrm{rge}(f)$.

Fact 15. ([M*2, Lemma 1.5]) Suppose $A, B \subseteq \mathcal{F}$, $A \subseteq B$, and for all $(\alpha, f) \in A$ and $(\beta, g) \in B \setminus A$

one has that $\alpha \leq \beta$. Suppose $\nu < \tau < \zeta < \kappa^+$. Then if $c_A(\tau,\zeta) < \kappa$ and $c_A(\tau,\zeta) \leq c_A(\zeta,\nu)$ one has that $c_B(\tau,\zeta) \leq c_B(\zeta,\nu)$.

Fact 16. ([M*2, Lemma 1.6] Suppose $A \subseteq \mathcal{F}$, $A = B \cup \{(\beta,g)\}$, and $\alpha \leq \beta$ for all $(\alpha,f) \in A$. Suppose $\nu < \tau < \zeta < \kappa^+$ and $\nu \in \mathrm{rge}(g)$ whenever $\tau \in \mathrm{rge}(g)$. Then $c_A(\nu,\tau) \leq c_A(\tau,\zeta)$ implies that $c_B(\nu,\tau) \leq c_B(\tau,\zeta)$.

Fact 17. ([M*2, Lemma 1.9]) Suppose that $A \subseteq \mathcal{F}$, $\alpha < \kappa$, $\nu < \tau < \kappa^+$ and $c_A(\nu,\tau) = \alpha$. Then there are witnesses $\langle \xi_i \mid i \leq k\rangle$ and $\langle (\alpha_i, f_i) \mid i < k\rangle$ to the fact that $c_A(\nu,\tau) = \alpha$ which are good.

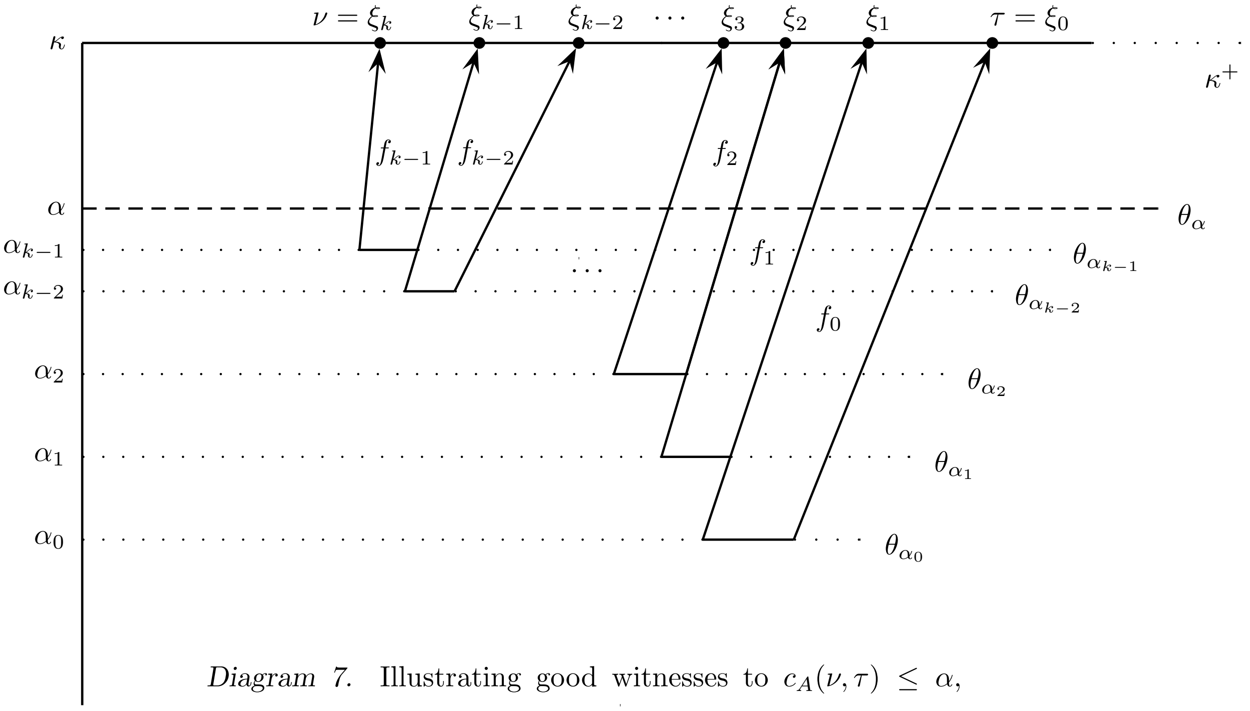

Diagram 7. Illustrating good witnesses to $c_A(\nu,\tau) \leq \alpha$, *cf.* Definition (12) and Fact (17), where $(\alpha_i, f_i) \in A$.

Fact 18. ([M*2, Lemma 1.11])

$$\forall A, B \subseteq \mathcal{F}\, \forall \tau < \zeta < \nu < \kappa^+ \big(A \subseteq B \longrightarrow (c_B(\tau,\zeta) \leq c_B(\zeta,\nu) \longrightarrow c_A(\tau,\zeta) \leq c_A(\zeta,\nu)\big).$$

Fact 19. ([M*2, Lemma 1.14]) Suppose $\xi < \zeta < \kappa^+$, $\alpha < \kappa$, $D \subseteq \bigcup\{\mathcal{F}_{\gamma\alpha} \mid \gamma < \alpha\}$ and (α, F), $(\alpha, G) \in \mathcal{F}$. Set $A = \{(\gamma, F \cdot g) \mid \exists \gamma < \alpha\ g \in D \cap \mathcal{F}_{\gamma\alpha}\}$, $B = \{(\gamma, G \cdot g) \mid \exists \gamma < \alpha\ g \in D \cap \mathcal{F}_{\gamma\alpha}\}$ and $E = A \cup B$. Suppose $c_E(\xi,\zeta) \leq \alpha$. If $\zeta \in \mathrm{rge}(F)$, then $c_E(\xi,\zeta) = c_A(\xi,\zeta)$, and if $\zeta \in \mathrm{rge}(G)$ then $c_E(\xi,\zeta) = c_B(\xi,\zeta)$.

Finally, one of the most important basic facts about the functions c_A is that they are subadditive.

Fact 20. ([M*2, Theorem 1.12] If $A \subseteq \mathcal{F}$ and $\nu < \tau < \zeta < \kappa^+$ then

(i) $$c_A(\nu, \tau) \leq \max(\{c_A(\nu, \tau), c_A(\tau, \zeta)\}),$$

(ii) $$c_A(\nu, \zeta) \leq \max(\{c_A(\nu, \tau), c_A(\tau, \zeta)\}).$$

Now suppose that $\mathbb{M}$ is stationary, and let $\mu = \kappa^-$ be the cardinal predecessor of κ. (Up until now everything mentioned was true for any regular κ, but the following definition as stated only makes sense for successor κ.)

Definition 21. Let λ be a regular cardinal greater than ω_2. Then $(\mathcal{N}, \epsilon) \prec (H_\lambda, \in)$ is a *good model* if $\overline{\overline{\mathcal{N}}} = \mu$, $\{\mathbb{M}, \mathcal{F}, c, \kappa, \kappa^+\} \cup \mu \subseteq \mathcal{N}$, $\mathcal{N}^{<\mu} \subseteq \mathcal{N}$, $\delta = \mathcal{N} \cap \kappa \in \kappa$, there is some $F \in \mathcal{F}_{\delta\kappa}$ such that $\mathrm{rge}(F) = \mathcal{N} \cap \kappa^+$, and for each $(\alpha, f) \in \mathcal{F}$ with $\alpha < \delta$ if there is some $f' \in \mathcal{F}_{\alpha\delta}$ such that $f = F \cdot f'$ then $(\alpha, f) \in \mathcal{N}$..

The following two observations about good models are useful in the next section.

Lemma 22. If $\mathcal{N}$ is good, $\alpha < \delta$, $g \in \mathcal{F}_{\alpha\kappa}$, $g' \in \mathcal{F}_{\alpha\delta}$, $g'' \in \mathcal{F}_{\delta\kappa}$ and $g = g'' \cdot g'$ then $h(g) = F \cdot g' \in \mathcal{F}_{\alpha\kappa} \cap \mathcal{N}$ (and so $(\alpha, h(g)) \in \mathcal{F} \cap \mathcal{N}$).

Proof. As $\mathcal{N}$ is good, if $\alpha < \delta$ and $g' \in \mathcal{F}_{\alpha\delta}$ then $F \cdot g' \in \mathcal{N}$.

Lemma 23. If $\mathcal{N}$ is good and $(\alpha, f) \in \mathcal{N}$ then $\exists f' \in \mathcal{F}_{\alpha\delta}$ $(f = F \cdot f')$.

Proof. If $(\alpha, f) \in \mathcal{N}$ then, firstly, $f \in \mathcal{N}$, and, secondly, $\alpha < \delta$, since $\delta = \mathcal{N} \cap \kappa$. Since $\mathbb{M} \in \mathcal{N}$ and $\alpha \in \mathcal{N}$ one has that $\theta_\alpha < \delta$, and hence that $\theta_\alpha \subseteq \mathcal{N}$. So if $\xi < \theta_\alpha$ then $f(\xi) \in \mathcal{N}$. Hence $\mathrm{rge}(f) \subseteq \kappa^+ \cap \mathcal{N} = \mathrm{rge}(F)$.

Now factor f as $k \cdot h$, where $(\delta, k) \in \mathcal{F}$ and $h \in \mathcal{F}_{\alpha\delta}$. Let $\overline{\theta} = \mathrm{ssup}(h``\theta_\alpha)$. Then there is some $\zeta \leq \theta_\delta$ such that $\mathrm{ssup}(F``\zeta) = \mathrm{ssup}(k``\overline{\theta})$. Consequently, by Fact (7), one has that $\zeta = \overline{\theta}$ and $F \restriction \overline{\theta} = k \restriction \overline{\theta}$. Thus one also has that $f = F \cdot h$.

However the main point of introducing the notion of a good model is in order to formulate the following definition and fact.

Definition 24. Let $\mathbb{P}$ be forcing notion with $\mathbb{P} \in H_\lambda$ for some regular cardinal λ. $\mathbb{P}$ is κ-$\mathbb{M}$-*proper* if there is some $x \in [H_\lambda]^\mu$ such that the following holds. Suppose $p \in \mathbb{P}$ and $\mathcal{N}$ is a good model with $\{\mathbb{P}, p\} \cup x \cup p \subseteq \mathcal{N}$. Then there is some $p^* \leq p$ which is $(\mathbb{P}, \mathcal{N})$-generic.

Fact 25. If $\mathbb{P}$ is κ-$\mathbb{M}$-proper then $\Vdash_{\mathbb{P}} \mu^+ = \kappa$.

§2. The general framework for proofs of κ-$\mathbb{M}$-properness.

Now I give the general framework of proofs of κ-$\mathbb{M}$-properness for the forcings discussed in this paper. In order to complete the proof of κ-$\mathbb{M}$-properness in each individual case one will have to complete the steps labeled 'Step i' for each i from A to D.

Suppose that $\mathbb{P}$ is a forcing and that conditions in $\mathbb{P}$ are of the form $p = (a_p, A_p, \ldots)$ where $a_p \in [\kappa^+]^{<\mu}$ is the *realm* of p and $A_p \in [\mathcal{F}]^{<\mu}$ is the *side condition part* of p.

Fix $p \in \mathbb{P}$. Suppose that $\mathcal{N} \prec H_{\kappa^{++}}$ is countable, $\mathbb{P}$, p, $\mathbb{M}$, $\mathcal{F}$, κ, $\kappa^+ \in \mathcal{N}$, p, $\mu \subseteq \mathcal{N}$, $\delta = \mathcal{N} \cap \kappa \in \kappa$ and that there is some $F \in \mathcal{F}_{\delta\omega_1}$ such that $F``\theta_\delta = \mathcal{N} \cap \kappa$.

The aim of the proofs will be to show that $(a_p, A_p \cup \{(\delta, F)\})$ is $(P, \mathcal{N})$-generic. The first step that needs to be carried out in the specific cases is

Step A. Prove that $(a_p, A_p \cup \{(\delta, F)\}, \ldots) \in \mathbb{P}$ and that $(a_p, A_p \cup \{(\delta, F)\}, \ldots) \leq p$.

If this can be done suppose that $\mathcal{D}$ is a dense, open subset of $\mathbb{P}$ with $\mathcal{D} \in \mathcal{N}$. Suppose also that $q \in \mathcal{D}$ and that $q \leq (a_p, A_p \cup \{(\delta, F)\}, \ldots)$. For technical reasons one also needs to be able to do the following

Step B. Show that one may without loss of generality assume that $c_{A_q}(\nu, \tau) < \kappa$ for all ν, $\tau \in a_q$.

Next let $q \restriction \mathcal{N} = q \cap \mathcal{N} = (a_q \cap \mathcal{N}, A_q \cap \mathcal{N}, \ldots)$.

Step C. Prove that $q \restriction \mathcal{N} \in \mathbb{P} \cap \mathcal{N}$ and that $q \leq q \restriction \mathcal{N}$.

If Step (C) can be carried out then next make the following long notational definition.

Notation. Let $\langle (\beta_i, f_i) \mid i < \chi \rangle$ enumerate $\langle (\alpha, f) \in A_q \mid \alpha < \delta \rangle$ for some $\chi < \mu$.

Let $Y_i = \mathrm{rge}(f_i) \cap \mathrm{rge}(h(f_i))$ for $i < \chi$.

Note that $\quad Y_i = \mathrm{rge}(f_i) \cap \mathrm{rge}(F) \quad \text{for all} \quad i < \chi \quad \text{as well.}$

Let $\quad \rho_i = \mathrm{ssup}(\{\rho < \theta_{\alpha_i} \mid f_i(\rho) = h(f_i)(\rho)\})$.

Then $\quad Y_i = \mathrm{rge}(f_i \restriction \rho_i) = \mathrm{rge}(\psi_{(\alpha_i,\rho_i),(\kappa,f_i(\rho_i))} \restriction \rho_i)$.

Let $\quad \beta^* = \mathrm{ssup}(\{\beta_i \mid i < \chi\})$.

Let $\quad C_q = \{c_{A_q}(\zeta,\xi) \mid \zeta, \xi \in a_q\} = c_{A_q}``[a_q]^2$.

Let $\quad e_q \in [\kappa]^{<\mu}$ be arbitrary, $e_{q\restriction\mathcal{N}} = e_q \cap \mathcal{N}$, and $\beta^\dagger = \max(\{\beta^*, \mathrm{ssup}(e_{q\restriction\mathcal{N}})\})$.

Write $c^q(\zeta,\xi)$ in place of $c_{A_q}(\zeta,\xi)$, and similarly for other conditions in $\mathbb{P}$, in order to save on subscripts!

Note that as $\overline{\overline{Y_i}}$ is an initial segment of $\mathrm{rge}(h(f_i)) \subseteq \mathrm{rge}(F) = \mathcal{N}$ for $i < \chi$ (by Fact (I.3)), one has that $Y_i \subseteq \mathcal{N}$ for each $i < \chi$. Note also that $Y_i \in \mathcal{N}$ and as $\mathcal{N}^{<\mu} \subseteq \mathcal{N}$ one has that $\langle Y_i \mid i < \chi \rangle \in \mathcal{N}$.

Let $\quad \{\xi_i \mid i < \lambda\} \quad \text{enumerate} \quad (a_q \setminus \mathcal{N}) \cap \sup(\mathcal{N} \cap \kappa^+)$.

Let $\quad \eta_i = \min(\mathcal{N} \setminus \xi_i) \quad \text{and} \quad \nu_i = \sup(\{\sup(Y_j \cap \eta_i) \mid j < \chi\}) \quad \text{for} \quad i < \lambda$.

Let $\phi(x)$ be the conjunction of the following.

(i) $\quad x \in \mathcal{D}$, $\quad$ (ii) $x \leq q \restriction \mathcal{N}$, $\quad$ and $\quad$ (iii) $\exists(\alpha^*, h^*) \in A_x \;\; \beta^\dagger < \alpha^*$ such that

(1) $\quad \forall(\beta, f) \in A_x \; \big(\beta < \beta^* \longrightarrow \exists i < \chi \; (\mathrm{rge}(f) \cap \mathrm{rge}(h^*) = Y_i)\big)$,

(2) $\quad \forall i < \chi \; \exists(\beta_i, f) \in A_x \; \big(\mathrm{rge}(f) \cap \mathrm{rge}(h^*) = Y_i \;\; \& \;\; \exists f'' \in \mathcal{F}_{\alpha^*\kappa} \; \exists f' \in \mathcal{F}_{\beta\alpha^*}$

$\qquad f = f'' \cdot f' \longrightarrow \mathrm{rge}(h^* \cdot f') \cap \mathrm{rge}(f) = Y_i\big)$,

(3) $\quad a_{q\restriction\mathcal{N}} \subseteq \mathrm{rge}(h^*)$,

(4) $\quad (a_x \setminus a_{q\restriction\mathcal{N}}) \cap \mathrm{rge}(h^*) = \emptyset$,

(5) $\quad e_{q\restriction\mathcal{N}} \subseteq e_x \;\; \& \;\; (e_x \setminus \beta^\dagger) \cap \alpha^* = \emptyset$,

(6) $\quad \forall\gamma \in C_x \; \gamma < \kappa$, $\quad$ and

Finally let $\phi'(x)$ be the conjunction of $\phi(x)$ with

(7) $\quad \forall i < \lambda \; \big(\nu_i, \, \eta_i \in \mathrm{rge}(h^*) \;\; \& \;\; \exists\zeta_i \in (\nu_i, \eta_i) \cap \mathrm{rge}(h^*)$

$\qquad\qquad \big(\text{there is a good model } \mathcal{M} \text{ such that } \mathrm{otp}([\nu_i, \zeta_i) \cap \mathrm{rge}(h^*)) \notin \mathcal{M}\big)\big)$.

(Clause (7) of ϕ' says in alternative notation that

$$\forall i < \lambda \left(\nu_i,\ \eta_i \in \mathrm{rge}(h^*)\ \ \&\ \ \exists\zeta_i \in (\nu_i, \eta_i) \cap \mathrm{rge}(h^*)\ \left(0 <^{\beta^\dagger} \mathrm{otp}([\nu_i, \zeta_i) \cap \mathrm{rge}(h^*))\right)\right).$$

This is because $\eta < \kappa$ is called a *β-number* if whenever $\mathcal{M}$ is a good submodel of $H_{\kappa^{++}}$ *and* $\beta \in \mathcal{M}$ then $\eta \in \mathcal{M} \cap \kappa$, and $\nu <^\beta \tau$, for ν, τ, and $\beta < \kappa$ means that $\nu + \eta < \tau$ for every if η which is a β-number.)

Then $H_{\kappa^{++}} \models$ "$\phi(q)$" and $H_{\kappa^{++}} \models$ "$\phi'(q)$". So by the elementarity of $\mathcal{N}$ in $H_{\kappa^{++}}$ there is some $s \in \mathcal{N}$ and $\mathcal{N} \models \phi(s)$. Let $(\alpha^*, h^*) \in \mathcal{N}$ be the witness to (iii) in the definition of ϕ for s in $\mathcal{N}$. Set $a_r = a_s \cup a_q$ and $A_r = A_s \cup A_q$.

Step D. Show that there is a condition r with realm a_r and side condition part A_r such that $r \in \mathbb{P}$ and $r \leq q,\ s$.

The pay-off of this abstract set-up is that it is proved in [M*2] that one now has the following properties which one hopes to use in order to complete Step (D).

[M*2, 4.12] $\forall\zeta_0 < \zeta_1 < \kappa^+ \left(\zeta_0,\ \zeta_1 \in a_s \longrightarrow c^s(\zeta_0, \zeta_1) = c^r(\zeta_0, \zeta_1) < \delta\right)$.

[M*2, 4.13] $\forall\xi < \zeta < \kappa^+ \left(\xi \in a_q \setminus a_s,\ \ \&\ \ \zeta \in a_s \longrightarrow \delta \leq c^r(\xi, \zeta)\right)$.

[M*2, 4.15] $\forall\sigma < \xi < \kappa^+ \left(\xi \in a_q\ \ \&\ \ \beta < \alpha^* \longrightarrow (c^r(\sigma, \xi) = \beta \longleftrightarrow c^q(\sigma, \xi) = \beta)\right)$.

[M*2, 4.16] $\forall\xi_0 < \xi_1 < \kappa^+,\ \forall\beta \in [\delta, \kappa)\ \left(c^q(\xi_0, \xi_1) \leq \beta \longleftrightarrow c^r(\xi_0, \xi_1) \leq \beta\right)$.

[M*2, 4.17] $\forall\xi < \zeta < \kappa^+ \left(\xi \in a_q \setminus a_s\ \ \&\ \ \zeta \in a_s \longrightarrow \delta \leq c^r(\xi, \zeta) = c^q(\xi, \zeta)\right)$.

[M*2, 4.19] $\forall\zeta < \xi < \kappa^+ \left(\zeta \in a_s \setminus a_q\ \ \&\ \ \xi \in a_q \longrightarrow \alpha^* \leq c^r(\zeta, \xi)\right)$.

[M*2, 4.21] $\forall\tau < \xi < \kappa^+ \left(\tau \in a_s \cap a_q\ \ \&\ \ \xi \in a_q \setminus a_s \longrightarrow (c^r(\tau, \xi) \leq \alpha^* \text{ or } c^r(\tau, \xi) \geq \delta)\right)$.

[M*2, 4.22] $\forall\zeta < \tau,\ \xi < \kappa^+ \left(\zeta \in a_s \setminus a_q\ \ \&\ \ \tau \in a_s \cap a_q\ \ \&\ \ \xi \in a_q \setminus a_s \longrightarrow c^r(\zeta, \tau) \leq c^r(\zeta, \xi)\right)$.

[M*2, 4.24] $\forall\zeta,\ \tau,\ \xi < \kappa^+ \left(\zeta \in a_s \setminus a_q\ \ \&\ \ \tau \in a_s \cap a_q\ \ \&\ \ \xi \in a_q \setminus a_s \longrightarrow c^r\{\zeta, \tau\} \leq c^r\{\zeta, \xi\}\right)$.

(Recall that if $\nu,\ \upsilon < \kappa^+$ then $c^r\{\nu, \upsilon\} = c^r(\nu, \upsilon)$ if $\nu < \upsilon$, and $= c^r(\upsilon, \nu)$ if $\upsilon < \nu$.)

[M*2, 4.27] $\forall\xi_0 < \xi_1 < \kappa^+ \left(\xi_0,\ \xi_1 \in a_q \setminus a_s \longrightarrow c^r(\xi_0, \xi_1) = c^q(\xi_0, \xi_1)\right)$.

§3. Scattered spaces of size μ^{++}.

Theorem 1. If there is a stationary $(\kappa, 1)$-simplified morass and κ is the successor of a regular cardinal μ with $2^{<\mu} \leq \kappa$ then there is a cardinal preserving forcing which adds a 0-dimensional, locally μ-Lindelöf μ-tight, scattered space of size κ^+.

In this section I show how to use κ-$\mathbb{M}$-proper forcing to add a space as described in the statement of Theorem (1). I recall the topological definition.

Definition 2. A topological space X is *μ-Lindelöf* if every open cover has a subcover of size *less than* μ. Thus ω-Lindeöf means compact and ω_1-Lindelöf means Lindelöf. X is locally μ-Lindelöf if every point has an open neighbourhood whose closure is μ-Lindelof. A space is 0-dimensional if its topology has a basis of sets which are both closed and open. A space X is *scattered* or *right-separated* if there is a well ordering $<_X$ of it such that for every $x \in X$ there is an open neighbourhood $\mathcal{O}$ of x such that $\mathcal{O} \cap \{y \in X \mid x <_X y\} = \emptyset$. A space X is μ-tight if for every $Y \subseteq X$ and x is in the closure of Y there is some $Z \in [Y]^{\leq \mu}$ such that x is in the closure of Z.

A space as described in Theorem (1) was added for $\mu = \omega$ by [BS] — although they worked in terms of superatomic Boolean algebras, Stone duality gives an immediate translation. It would be nice to be able to prove that one could force to add a 0-dimensional, locally compact, countably tight, scattered space of size μ^{++}, however it seems unlikely that one will be able to do this with conditions whose working parts are not finite. I sketch why this is so at the end of the section.

Definition 3. Define a partial order $\mathbb{P}^-$ with elements of the form $p = (a_p, \leq_p, i_p)$, where $a_p \in [\kappa^+]^{<\mu}$, $\leq_p$ is a partial order on a_p such that $\alpha \leq_p \beta$ implies $\alpha \leq \beta$, and $i_p : [a_p]^2 \longrightarrow [a_p]^{<\mu}$ is defined by $i_p(\alpha, \beta) = \left\{\gamma \in a_p \,\middle|\, \gamma \leq \alpha,\ \beta\ \ \&\ \ \forall \varepsilon \in a_p\ \left(\neg(\gamma <_p \varepsilon \leq_p \alpha,\ \beta)\right)\right\}$ and is included in the condition p purely for notational purposes. (Here $\gamma <_p \varepsilon$ abbreviates $\gamma \leq_p \varepsilon\ \ \&\ \ \gamma \neq \varepsilon$, as usual). The order on $\mathbb{P}^-$ is $p' \leq p$ if $a_p \subseteq a_{p'}$, $\leq_p = \leq_{p'} \restriction a_p \times a_p$ and $i_p = i_{p'} \restriction [a_p]^2$.

Of course one may equally well write $p = (a_p, \leq_p, i_p)$ as $p = (a_p, h_p, i_p)$ where $h_p : a_p \times a_p \longrightarrow 2$ is a function such that $\forall \alpha \in a_p\ h_p(\alpha, \alpha) = 1$; $\forall \alpha,\ \beta \in a_p\ (\alpha < \beta \longrightarrow h_p(\alpha, \beta) = 0$; and $\forall \alpha,\ \beta,\ \gamma \in a_p\ (h_p(\alpha, \beta) = 1\ \ \&\ \ h_p(\beta, \gamma) = 1 \longrightarrow h_p(\alpha, \gamma) = 1)$, that is, that h_p satisfies the requirements to be the characteristic function of a partial order on a_p.

Note that $\mathbb{P}^-$ does not have the κ-chain condition: if one takes κ many conditions and applies the Δ-system lemma to their 'a's then it may still be the case that no two of the 'i's agree on their restriction to the root of the Δ-system.

Definition 4. Define a forcing notion $\mathbb{P}$ with conditions $p = (a_p, \leq_p, i_p, A_p)$ where $(a_p, \leq_p, i_p) \in \mathbb{P}^-$, $A_p \in \mathcal{F}$ and, writing c^p for c_{A_p},

$$\forall\alpha,\ \beta \in a_p\ (\alpha < \beta \longrightarrow i_p(\alpha, \beta) \subseteq \{\, \gamma \leq \alpha \,|\, c^p(\gamma, \alpha) \leq c^p(\alpha, \beta) \,\}).$$

The order is the product of the order on $\mathbb{P}^-$ and reverse inclusion, *i.e.* $p' \leq p$ if $a_p \subseteq a_{p'}$, $\leq_p = \leq_{p'} \restriction a_p \times a_p$, $i_p = i_{p'} \restriction [a_p]^2$ and $A_{p'} = A_p$. a_p is the *realm* of p and A_p is the *side condition part* of p.

The intention is that if G is $\mathbb{P}$-generic one will take $B_\alpha = \{\, \beta \leq \alpha \,|\, \exists p \in G\ (\beta,\ \alpha \in a_p\ \ \&\ \ \beta \leq_p \alpha) \,\}$. Taking $\mathcal{B} = \{\, B_\alpha \setminus \bigcup\{\, B_\beta \,|\, \beta \in b \,\} \,|\, \alpha < \kappa^+\ \ \&\ \ b \in [\alpha]^{<\mu} \,\}$ will give a basis for a topology on κ^+ which satisfies the theorem provided that cardinals are preserved. I prove the topological properties after showing that cardinals are indeed preserved by the forcing.

Proposition 5. $\mathbb{P}$ is κ-$\mathbb{M}$-proper.

Proof. So suppose that $p \in \mathbb{P}$ and that $\mathcal{N}$ is as in the general framework but for this specific $\mathbb{P}$.

Step A. Checking that $p' = (a_p, \leq_p, i_p, A_p \cup \{\, (\delta, F) \,\})$ is a condition and is stronger than p.

Proof. Observe that $(\alpha, f) \in \mathcal{N}$ for each $(\alpha, f) \in A_p$, since $p \subseteq \mathcal{N}$. So for each $(\alpha, f) \in A_p$ one has, by (i) of the properties of $\mathcal{N}$ given in §2, that there is some $f' \in \mathcal{F}_{\alpha\beta}$ such that $f = F \cdot f'$. Suppose that $\alpha,\ \beta,\ \gamma \in a_p,\ \gamma < \alpha < \beta$ and $c^p(\gamma, \alpha) \leq c^p(\alpha, \beta)$. As $a_p \subseteq \mathcal{N} \cap \kappa^+ = \mathrm{rge}(F)$ one has that $c^{p'}(\gamma, \alpha) \leq c^{p'}(\alpha, \beta)$ by Fact (1.16). This shows that $i_{p'}(\alpha, \beta) \subseteq \{\gamma < \alpha \,|\, c^{p'}(\gamma, \alpha) \leq c^{p'}(\alpha, \beta)\}$. Hence $p' \in \mathbb{P}$.

Suppose that $\mathcal{D}$ is a dense, open subset of $\mathbb{P}$, $\mathcal{D} \in \mathcal{N}$ and $q \in \mathcal{D}$ where $q \leq p'$.

Step B. Show that without loss of generality one may assume that $c^q(\nu, \tau) < \kappa$ for all $\nu,\ \tau \in a_q$.

Proof. For if q does not satisfy this one may simply add to A_q a map (β, f) such that $a_q \subseteq \mathrm{rge}(f)$ and $\alpha \leq \beta$ for all $(\alpha, g) \in A_q$ by applying Fact (1.5). By Fact (1.16), this augmentation of q is still a condition, it is stronger than q by the definition of $\leq_{\mathbb{P}}$, and is in $\mathcal{D}$ because $\mathcal{D}$ is open.

Let $q \restriction \mathcal{N} = q \cap \mathcal{N}$.

Step C. Show that $q \restriction \mathcal{N} \in \mathbb{P} \cap \mathcal{N}$ and $q \leq q \restriction \mathcal{N}$.

Lemma 6. If τ_0, $\tau_1 \in a_q \cap \mathcal{N}$ then $i_q(\tau_0, \tau_1) \subseteq \mathcal{N}$. And if $\gamma < \tau_0 < \tau_1$ and $c^q(\gamma, \tau_0) \leq c^q(\tau_0, \tau_1)$ then $c^{q\restriction\mathcal{N}}(\gamma, \tau_0) \leq c^{q\restriction\mathcal{N}}(\tau_0, \tau_1)$.

Proof. As τ_0, $\tau_1 \in \mathcal{N} = \mathrm{rge}(F)$, one has that $c^q(\tau_0, \tau_1) \leq \delta$. If $\gamma \in i_q(\tau_0, \tau_1)$ then $c^q(\gamma, \tau_0) \leq \delta$ by the constraint on i_q in the definition of what it is to be a condition in $\mathbb{P}$. So by Fact (1.14) one has that $\gamma \in \mathrm{rge}(F)$. By Fact (1.18), for every $\gamma < \tau_0$, if $c^q(\gamma, \tau_0) \leq c^q(\tau_0, \tau_1)$ one has that $c^{q\restriction\mathcal{N}}(\gamma, \tau_0) \leq c^{q\restriction\mathcal{N}}(\tau_0, \tau_1)$.

Corollary 7. $q \leq q \restriction \mathcal{N} \in \mathbb{P} \cap \mathcal{N}$.

Proof. If τ_0, $\tau_1 \in a_q \cap \mathcal{N}$ and $\gamma \in i_q(\tau_0, \tau_1)$ then, by the first half of Lemma (6), $\gamma \in a_{q\restriction N}$ and so $i_{q\restriction\mathcal{N}}(\tau_0, \tau_1) = i_q(\tau_0, \tau_1)$. So $i_{q\restriction\mathcal{N}}(\tau_0, \tau_1) = i_q(\tau_0, \tau_1) \subseteq \{\gamma \leq \tau_0 \mid c^q(\gamma, \tau_0) \leq c^q(\gamma, \tau_0)\} \subseteq \{\gamma \leq \tau_0 \mid c^{q\restriction\mathcal{N}}(\gamma, \tau_0) \leq c^{q\restriction\mathcal{N}}(\gamma, \tau_0)\}$, by the second half of Lemma (6). Hence as far as the pair τ_0, τ_1 are concerned, $q \restriction \mathcal{N}$ satisfies the requirement for being a condition, and q and $q \restriction \mathcal{N}$ satisfy the requirement for q to be a stronger condition than $q \restriction \mathcal{N}$.

Now obtain s as in §2 using b_q as the arbitrary set e_q in the section of Notation in §2.

Step D. Define some r with realm $a_s \cup a_q$ and side condition part $A_s \cup A_q$. Show that $r \in \mathbb{P}$ and that $r \leq s$, q.

Proof. I define r as follows. Let $a_r = a_s \cup a_q$ and $A_r = A_s \cup A_q$. Let $\alpha \leq_r \beta$ if $\alpha \leq_q \beta$ for $\alpha \leq_s \beta$ or $\exists\gamma\ (\alpha \leq_s \gamma \leq_q \beta)$ or $\exists\gamma\ (\alpha \leq_q \gamma \leq_s \beta)$.

Lemma 8. If $\{\alpha, \gamma\} \in [a_q]^2$, $\beta \in a_s$ and ε, $\nu \in a_s \cap a_q$ are such that $\alpha \leq_q \varepsilon \leq_s \beta \leq_s \nu \leq_q \gamma$ then $\alpha \leq_q \gamma$. The same conclusion also holds with s and q exchanged.

Proof. One has that $\alpha \leq_q \varepsilon \leq_s \nu \leq_q \gamma$ by the transitivity of $\leq_s$. Moreover, $\varepsilon \leq_q \nu$ since $\leq_q \restriction a_{q\restriction\mathcal{N}} = \leq_s \restriction a_{q\restriction\mathcal{N}}$, so $\alpha \leq_q \gamma$ by the transitivity of $\leq_q$. The proof with q and s exchanged is identical.

Corollary 9. $\leq_r \restriction a_q \times a_q = \leq_q$ and $\leq_r \restriction a_s \times a_s = \leq_s$.

Lemma 10. $i_r \restriction [a_q]^2 = i_q$ and $i_r \restriction [a_s]^2 = i_s$.

Proof. Suppose $\xi_0,\ \xi_1 \in a_q$ and $\nu \leq_r \xi_0,\ \xi_1$. If $\nu \in a_q$ then $\nu \leq_q \xi_0,\ \xi_1$, by Corollary (9), so there is some $\xi \in i_q(\xi_0, \xi_1)$ such that $\nu \leq_q \xi$. If $\nu \in a_s \setminus a_q$ then there are $\tau_0,\ \tau_1 \in a_s \cap a_q$ such that $\nu \leq_s \tau_0 \leq_q \xi_0$ and $\nu \leq_s \tau_1 \leq_q \xi_1$. Let $\tau \in i_s(\tau_0, \tau_1)$ be such that $\nu \leq_s \tau$. Now, as $q,\ s \leq q \restriction \mathcal{N}$ one has that $i_q(\tau_0, \tau_1) = i_{q\restriction\mathcal{N}}(\tau_0, \tau_1) = i_s(\tau_0, \tau_1)$. So $\tau \in i_q(\tau_0, \tau_1)$. Let $\xi \in i_q(\xi_0, \xi_1)$ be such that $\tau \leq_q \xi$. Then $\nu \leq_r \tau \leq_r \xi \leq_r \xi_0,\ \xi_1$, so $\nu \leq_r \xi \in i_q(\xi_0, \xi_1)$. This shows that $i_r \restriction [a_q]^2 = i_q$. The argument that $i_r \restriction [a_s]^2 = i_s$ is identical with the rôles of s and q swapped.

Lemma 11. $\leq_r$ is a partial order on $a_r \times a_r$.

Proof. $\leq_r$ is clearly reflexive and anti-symmetric, so only transitiviy needs to be shown. Suppose $\alpha \leq_r \beta \leq_r \gamma$. If $\{\alpha, \beta, \gamma\} \in [a_q]^3 \cup [a_s]^3$ then $\alpha \leq_r \gamma$. If $\{\alpha, \gamma\} \in [a_q]^2$ and $\beta \in a_s$ then $\alpha \leq_q \gamma$ by Lemma (8). If $\{\alpha, \beta\} \in [a_q]^2$ and $\gamma \in a_s$ let $\varepsilon \in a_s \cap a_q$ be such that $\alpha \leq_q \beta \leq_q \varepsilon \leq_s \gamma$. Then $\alpha \leq_q \varepsilon \leq_s \gamma$ by the transitivity of $\leq_q$. The argument in each of the other cases is identical to one of these latter two.

The arguments for Lemmas (8), (10) and (11) are essentially the same as analgous arguments used in [BS, §8] in the proof that the subcollection of $\mathbb{P}^-$ (when $\kappa = \omega_1$) consisting of those conditions p with $i_p(\alpha, \beta) \subseteq \{\gamma \leq \alpha \,|\, c(\gamma, \alpha) \leq c(\alpha, \beta)\}$ for all $\alpha,\ \beta \in a_p$, under the suborder of $\leq_{\mathbb{P}^-}$ such that that i remains fixed between a condition and a stronger one, has the ccc. Similarly to that proof, as a consequence of Lemmas (8), (10) and (11) it suffices to show that $r \in \mathbb{P}$ in order to show that q and s are compatible in $\mathbb{P}$. Here this will show that $\mathbb{P}$ is $\mathbb{M}$-proper. What needs to be done in order to complete this is to show that if $\alpha,\ \beta \in a_r$ with $\alpha < \beta$ and $\gamma \in i_r(\alpha, \beta)$ then $c^r(\gamma, \alpha) \leq c^r(\alpha, \beta)$. But the proof of this is markedly different from (and considerably harder than) the analgous proof in [BS, §8]. In conformity with the usage in [M*2, §4], I try to use τ, possibly adorned with subscripts, for variables in $a_s \cap a_q$, ζ for variables in a_s and ξ for variables in a_q. I remind the reader that the facts [M*2, 4.xx] from [M*2] cited below are listed at the end of §2 of this paper.

Proposition 12. $r \in \mathbb{P}$.

Proof.

Case A. ξ_0, $\xi_1 \in a_q$ and $\xi_0 < \xi_1$. Lemma (10) shows that $i_r(\xi_0, \xi_1) = i_q(\xi_0, \xi_1)$. So what needs to be shown is that if $\xi \in i_q(\xi_0, \xi_1)$ then $c^r(\xi, \xi_0) \leq c^r(\xi_0, \xi_1)$. Suppose that $\xi \in i_q(\xi_0, \xi_1)$.

Subcase A.i. ξ_0, $\xi_1 \in a_q \cap a_s = a_{q \restriction \mathcal{N}}$. If $\xi \in i_q(\xi_0, \xi_1)$ then $\xi \in a_q \cap \mathcal{N} = a_q \cap a_s$, by Lemma (6) and clauses (iii.3) and (iii.4) of the definition of ϕ for q, respectively. Thus $\xi \in i_{q \restriction \mathcal{N}}(\xi_0, \xi_1)$. As $s \leq q \restriction \mathcal{N}$ one has that $\xi \in i_s(\xi_0, \xi_1)$, and hence that $c^s(\xi, \xi_0) \leq c^s(\xi_0, \xi_1)$. By [M*2, 4.12] one has that $c^s(\xi, \xi_0) = c^r(\xi, \xi_0)$ and $c^s(\xi_0, \xi_1) = c^r(\xi_0, \xi_1)$. Thus $c^r(\xi, \xi_0) \leq c^r(\xi_0, \xi_1)$.

Subcase A.ii. ξ_0, $\xi_1 \in a_q \setminus a_s$. By Fact (1.13) one has that $c^r(\xi, \xi_0) \leq c^q(\xi, \xi_0)$, and one also has that $c^q(\xi, \xi_0) \leq c^q(\xi_0, \xi_1)$ since $q \in \mathbb{P}$ and $\xi \in i_q(\xi_0, \xi_1)$. Moreover, $c^q(\xi_0, \xi_1) = c^r(\xi_0, \xi_1)$ by [M*2, 4.27]. Hence $c^r(\xi, \xi_0) \leq c^r(\xi_0, \xi_1)$.

Subcase A.iii. $\xi_0 \in a_q \setminus a_s$ and $\xi_1 \in a_q \cap a_s = a_{q \restriction \mathcal{N}}$. One has $c^r(\xi, \xi_0) \leq c^q(\xi, \xi_0)$ by Fact (1.13), again, and $c^q(\xi, \xi_0) \leq c^q(\xi_0, \xi_1)$ since $q \in \mathbb{P}$ and $\xi \in i_q(\xi_0, \xi_1)$. One also has that $\delta \leq c^r(\xi_0, \xi_1) = c^q(\xi_0, \xi_1)$ by [M*2, 4.17]. Hence $c^r(\xi, \xi_0) \leq c^r(\xi_0, \xi_1)$.

Subcase A.iv. $\xi_0 \in a_q \cap a_s$ and $\xi_1 \in a_q \setminus a_s$. If $\delta \leq c^r(\xi_0, \xi_1)$ then $c^r(\xi_0, \xi_1) = c^q(\xi_0, \xi_1)$ by [M*2, 4.16], when $c^r(\xi, \xi_0) \leq c^r(\xi_0, \xi_1)$ as in Cases (A.ii) and (A.iii) again. Otherwise, $c^r(\xi_0, \xi_1) \leq \alpha^*$ by [M*2, 4.21]. If $c^r(\xi_0, \xi_1) < \alpha^*$ then $c^r(\xi_0, \xi_1) = c^q(\xi_0, \xi_1)$ by [M*2, 4.15], and $c^r(\xi, \xi_0) \leq c^r(\xi_0, \xi_1)$ as in Cases (A.ii) and (A.iii) once more. The remaining possibility is that $c^r(\xi_0, \xi_1) = \alpha^*$ and $c^q(\xi_0, \xi_1) = \delta$. As $\xi \in a_q$ one has that $c^q(\xi, \xi_0) \leq c^q(\xi, \xi_1) = \delta$. By Fact (1.14) one has that $\xi \in \mathrm{rge}(F) = \mathcal{N} \cap \kappa^+$, so that $\xi \in a_q \cap a_s$. By clause (iii.3) of the definition of ϕ for s one then has that ξ, $\xi_0 \in \mathrm{rge}(h^*)$, so that $c^r(\xi, \xi_0) \leq \alpha^*$, as required.

Case B. ζ_0, $\zeta_1 \in a_s$ and $\zeta_0 < \zeta_1$. Lemma (10) shows that $i_r(\zeta_0, \zeta_1) = i_s(\zeta_0, \zeta_1)$. So what needs to be shown is that if $\zeta \in i_s(\zeta_0, \zeta_1)$ then $c^r(\zeta, \zeta_0) \leq c^r(\zeta_0, \zeta_1)$. So suppose that $\zeta \in i_q(\zeta_0, \zeta_1)$. As previously, one has that $c^r(\zeta, \zeta_0) \leq c^s(\zeta, \zeta_0)$ by Fact (1.13), and that $c^s(\zeta, \zeta_0) \leq c^s(\zeta_0, \zeta_1)$ since $s \in \mathbb{P}$ and $\zeta \in i_s(\zeta_0, \zeta_1)$. But [M*2, 4.12] shows that $c^r(\zeta_0, \zeta_1) = c^s(\zeta_0, \zeta_1)$. So $c^r(\zeta, \zeta_0) \leq c^r(\zeta_0, \zeta_1)$ as required.

Case C. $\zeta \in a_s \setminus a_q$ and $\xi \in a_q \setminus a_s$. If $\zeta \leq_r \xi$ then $i_r(\zeta, \xi) = \{\zeta\}$ and, clearly, $c^r(\zeta, \zeta) \leq c^r(\zeta, \xi)$. Similarly if $\xi \leq_r \zeta$ then $i_r(\xi, \zeta) = \{\xi\}$ and $c^r(\xi, \xi) \leq c^r(\xi, \zeta)$. So suppose that $\zeta \not\leq_r \xi$ and $\xi \not\leq_r \zeta$.

Case C.i. $\nu \in a_s$ and $\nu \leq_r \xi$, ζ. Then, by the definition of $\leq_r$ there is some $\tau \in a_s \cap a_q$ such that $\nu \leq_s \tau \leq_q \xi$. Let $\eta \in i_s(\tau, \zeta)$ be such that $\nu \leq_s \eta$. Note that $\eta \in i_r(\tau, \zeta)$ by Lemma (10). I shall show that $c^r(\eta,\zeta) \le c^r(\zeta,\xi)$ if $\zeta < \xi$ and $c^r(\eta,\zeta) \le c^r(\xi,\zeta)$ if $\xi < \zeta$. The proof breaks into three cases.

Case C.i.a. $\eta < \xi < \zeta$. Then $c^r(\eta,\zeta) < \delta \le c^r(\xi,\zeta)$, by [M*2, 4.12] and [M*2, 4.17] respectively. Hence $c^r(\eta,\xi) \le c^r(\xi,\zeta)$, by the subadditivity of c^r.

Case C.i.b. $\eta \le \tau < \zeta < \xi$. Then $c^s(\eta,\tau) \le c^s(\tau,\zeta)$ since s is a condition in $\mathbb{P}$ and $\eta \in i_s(\tau,\zeta)$. So $c^r(\eta,\tau)$, $c^r(\eta,\zeta) \le c^r(\tau,\zeta)$, by [M*2, 4.12] and the subadditivity of c^r. By [M*2, 4.24] one has that $c^r(\tau,\zeta) \le c^r(\zeta,\xi)$. Hence $c^r(\eta,\zeta) \le c^r(\zeta,\xi)$.

Case C.i.c. $\eta \le \zeta < \tau < \xi$. Then $c^s(\eta,\tau) \le c^s(\zeta,\tau)$ since s is a condition in $\mathbb{P}$, $\eta \in i_s(\tau.\zeta)$ and by the subadditivity of c^s. So $c^r(\eta,\tau)$, $c^r(\eta,\zeta) \le c^r(\zeta,\tau)$, by [M*2, 4.12] and the subadditivity of c^r. By [M*2, 4.22] one has that $c^r(\zeta,\tau) \le c^r(\zeta,\xi)$. Hence $c^r(\eta,\zeta) \le c^r(\zeta,\xi)$, again.

Case C.ii. $\nu \in a_q \setminus a_s$ and $\nu \le_r \zeta, \xi$. By the definition of $\le_r$ there is some $\tau \in a_s \cap a_q$ such that $\nu \le_q \tau \le_s \zeta$. Let $\eta \in i_q(\tau,\xi)$ be such that $\nu \le_q \eta$. I shall show that $c^r(\eta,\zeta) \le c^r(\zeta,\xi)$ if $\zeta < \xi$ and $c^r(\eta,\xi) \le c^r(\xi,\zeta)$ if $\xi < \zeta$. The argument again splits into several cases.

Case C.ii.a. $\xi < \tau$. One has that $c^q(\eta,\xi) \le c^q(\xi,\tau)$, since $q \in \mathbb{P}$ and $\eta \in i_q(\xi,\tau)$, and hence $c^r(\eta,\xi) \le c^q(\eta,\xi) \le c^q(\xi,\tau) = c^r(\xi,\tau) \ge \delta$, the first inequality holding by Fact (1.3), and the equality and last inequality holding by [M*2, 4.17]. Now [M*2, 4.12] shows that $c^r(\tau,\zeta) < \delta$, so $c^r(\xi,\zeta) = c^r(\xi,\tau)$ by the subadditivity of c^r. Hence $c^r(\eta,\xi) \le c^r(\xi,\zeta)$.

Case C.ii.b. $\tau < \xi$. By Case (A) of this proof one has that $c^r(\eta,\tau) \le c^r(\tau,\xi)$.

Case C.ii.b.1. $\zeta < \xi$. Then $c^r(\tau,\xi)$, $c^r(\tau,\zeta) \le c^r(\zeta,\xi)$ by [M*2, 4.24]. One also has that $c^r(\eta,\zeta) \le \max(\{c^r(\eta,\tau), c^r(\tau,\zeta)\}$, by the subadditivity of c^r. Hence $c^r(\eta,\zeta) \le c^r(\tau,\zeta) \le c^r(\zeta,\xi)$ or $c^r(\eta,\zeta) \le c^r(\eta,\tau) \le c^r(\tau,\xi) \le c^r(\zeta,\xi)$, giving in either case that $c^r(\eta,\zeta) \le c^r(\zeta,\xi)$.

Case C.ii.b.2. $\xi < \zeta$ and $\eta \in a_s \cap a_q$. Then $c^r(\eta,\zeta) = c^s(\eta,\zeta) < \delta$ by [M*2, 4.12], and $\delta \le c^r(\xi,\zeta)$ by [M*2, 4.17]. Hence $c^r(\eta,\xi) \le c^r(\xi,\zeta)$, by the subadditivity of c^r.

Case C.ii.b.3. $\xi < \zeta$ and $\eta \in a_q \setminus a_s$. Then one has that $\delta \le c^r(\eta,\tau)$ by [M*2, 4.17]. As $c^r(\eta,\tau) \le c^r(\tau,\xi)$ (since this is a subcase of Case (C.ii.b)), this gives that $\delta \le c^r(\tau,\xi)$. Now $c^r(\tau,\zeta) < \delta$ by [M*2, 4.12], and so $c^r(\tau,\xi) \le c^r(\xi,\zeta)$ by the subaditivity of c^r. Hence $c^r(\eta,\xi) \le \max(\{c^r(\eta,\tau), c^r(\tau,\xi)\}) = c^r(\tau,\xi) \le c^r(\xi,\zeta)$.

CASE C.(*resumé*) The conclusions of subcases Case (C.i) and (C.ii) establish that if $\nu \leq_r \zeta$, ξ there is some η such that $\nu \leq_r \leq_r \eta \leq_r \zeta$, ξ and such that $c^r(\eta,\zeta) \leq c^r(\zeta,\xi)$ if $\zeta < \xi$, while $c^r(\eta,\xi) \leq c^r(\xi,\zeta)$ if $\xi < \zeta$. As a_r is of size $< \mu$ this shows that if $\eta \in i_r(\xi,\zeta)$ then $c^r(\eta,\zeta) \leq c^r(\zeta,\xi)$ if $\zeta < \xi$, while $c^r(\eta,\xi) \leq c^r(\xi,\zeta)$ if $\xi < \zeta$, as required.

Hence Cases (A), (B) and (C) taken together show that if α, $\beta \in a_r$ with $\alpha < \beta$ and $\gamma \in i_r(\alpha,\beta)$ then $c^r(\gamma,\alpha) \leq c^r(\alpha,\beta)$, and so that $r \in \mathbb{P}$, thus concluding the proof Proposition (12).

This shows that q and s are compatible and hence $\mathbb{P}$ is κ-$\mathbb{M}$-proper, concluding the proof of Proposition (5). Hence forcing with $\mathbb{P}$ preserves κ.

One great advantage of κ-$\mathbb{M}$-proper forcing over proper forcing is that one has a reasonable chance of preserving κ^+ (and all greater cardinals) as well as κ!

Proposition 13. $\mathbb{P}$ has the κ^+-cc.

Proof. Let $\{ p'_\nu \mid \nu < \kappa^+ \}$ be a subset of $\mathbb{P}$ of size κ^+. Extend each p'_ν in the original collection one is given to a stronger condition p_ν by adding a single map to $A_{p'_\nu}$ if necessary in order to ensure that there is a map $(\alpha_\nu, F_\nu) \in A_{p_\nu}$ such that for all $(\beta, f) \in A_{p_\nu}$ one has $\beta \leq \alpha$ and there is some $g \in \mathcal{F}_{\beta\alpha}$ such that $f = F \cdot g$ and such that $a_p \subseteq \mathrm{rge}(F)$. This is possible by Fact (1.5), as in the argument in the paragraph two prior to Lemma (6) above where q is introduced and then assumed to have this property. If one can find η, $\nu < \delta$ such that p_η and p_ν are compatible then, clearly, one will have that p'_η and p'_ν are compatible as well.

By thinning using the Δ-system lemma if necessary one may as well assume that there is some $a \in [\kappa^+]^{<\mu}$ such that a is an initial segment of each a_{p_ν}. One may as well also assume that if $\eta < \nu < \kappa^+$ then $\max(a_{p_\eta}) < \min(a_{p_\nu} \setminus a)$, and there is an order preserving isomorphism from $(a_{p_\eta}, \leq_{p_\eta})$ to $(a_{p_\nu}, \leq_{p_\nu})$. Moreover, one may as well assume that there is some $\alpha < \kappa$, some $A \in [\bigcup \{ \{\beta\} \times \mathcal{F}_{\beta\alpha} \mid \beta < \alpha \}]^{<\mu}$ and for each $\nu < \kappa^+$ there is a map $F_\nu \in \mathcal{F}_{\alpha\kappa}$ such that $A_{p_\nu} = \{ (\beta, F_\nu \cdot h) \mid (\beta, h) \in A \} \cup \{ (\alpha, F_\nu) \}$ and such that $F_\nu \cdot F_\eta^{-1} \restriction a_{p_\eta}$ is the order preserving isomorphism from $(a_{p_\eta}, \leq_{p_\eta})$ to $(a_{p_\nu}, \leq_{p_\nu})$. This is possible using the directedness of $\mathcal{F}$ and then applying the Δ-system lemma.

Clearly for each $\nu < \kappa^+$ and $\xi < \zeta < \kappa^+$ one has that $c^{p_\nu}(\xi,\zeta) \leq \alpha$ if and only if $c^{p_\nu}(\xi,\zeta) < \kappa$. Let $\eta < \nu < \kappa^+$ and set $a_r = a_{p_\eta} \cup a_{p_\eta}$, $\leq_r = \leq_{p_\eta} \cup \leq_{p_\eta}$ and $A_r = A_{p_\eta} \cup A_{p_\eta}$. Note that $i_r(\zeta_0,\zeta_1) = i_{p_\eta}$

if ζ_0, $\zeta_1 \in a_{p_\eta}$ and $i_r(\xi_0, \xi_1) = i_{p_\nu}$ if ξ_0, $\xi_1 \in a_{p_\nu}$. It is also clear that if $\xi < \zeta < \kappa^+$ then $c^r(\xi, \zeta) \leq \alpha$ if and only if $c^r(\xi, \zeta) < \kappa$. By Fact (1.19), one has that if $\zeta \in \mathrm{rge}(F_\eta)$ then $c^r(\xi, \zeta) = c^{p_\eta}(\xi, \zeta)$ and if $\zeta \in \mathrm{rge}(F_\nu)$ then $c^r(\xi, \zeta) = c^{p_\nu}(\xi, \zeta)$.

Claim 14. If $\xi \in a_{p_\eta} \setminus a_{p_\nu}$ then $\xi \notin \mathrm{rge}(F_\nu)$.

Proof. If $\xi \in \mathrm{rge}(F_\nu)$ then $\xi \in \mathrm{rge}(F_\nu) \cap \mathrm{rge}(F_\eta)$ since $F_\nu \cdot F_\eta^{-1} \restriction a_{p_\eta}$ is the order preserving isomorphism from a_{p_η} to a_{p_ν}. But then one has that $F_\nu(\xi_\alpha) = \xi = F_\eta(\xi_\alpha)$ by Fact (1.3), so $F_\nu \cdot F_\eta^{-1}(\xi) = \xi$ and hence $\xi \in a_{p_\nu} \cap a_{p_\eta}$, a contradiction.

It is now immediate that if $\xi \in a_{p_\eta} \setminus a_{p_\nu}$ and $\zeta \in a_{p_\nu} \setminus a_{p_\eta}$ then $\xi \notin \mathrm{rge}(F_\nu)$, by Claim (14), so that $c^{p_\nu}(\xi, \zeta) = \kappa$, and thus $c^r(\xi, \zeta) = \kappa$ from Claim (14). Hence for such ξ, ζ one has that $i_r(\xi, \zeta) \subseteq \{\, \tau \leq \xi \mid c^r(\tau, \xi) \leq c^r(\xi, \zeta) \,\}$.

By Claim (14), as well, if ξ_0, $\xi_1 \in a_{p_\eta}$ then $c^r(\xi_0, \xi_1) = c^{p_\eta}(\xi_0, \xi_1)$; and so if $\xi_0 < \xi_1$ one has that $i_r(\xi_0, \xi_1) \subseteq \{\, \tau \leq \xi_0 \mid e^{p_\eta}(\tau, \xi_0) \leq e^{p_\eta}(\xi_0, \xi_1) \,\} = \{\, \tau \leq \xi_0 \mid e^r(\tau, \xi_0) \leq e^r(\xi_0, \xi_1) \,\}$. (One gets equality rather than just the necessary inclusion by applying Claim (14) again for $\tau < \xi_0 \in \mathrm{rge}(F_\eta)$.) Similarly, if ζ_0, $\zeta_1 \in a_{p_\nu}$ then $c^r(\zeta_0, \zeta_1) = c^{p_\nu}(\zeta_0, \zeta_1)$. Thus $i_r(\zeta_0, \zeta_1) \subseteq \{\, \tau \leq \zeta_0 \mid e^r(\tau, \zeta_0) \leq e^r(\zeta_0, \zeta_1) \,\}$ if $\zeta_0 < \zeta_1$.

Hence $r \in \mathbb{P}$, and thus $r \leq_{\mathbb{P}} p_\eta$, p_ν. So $\mathbb{P}$ does have the κ^+-cc.

Thus cardinals above μ are preserved by forcing with $\mathbb{P}$. The next proposition shows that cardinals up to μ are also preserved.

Proposition 15. $\mathbb{P}$ is μ-closed.

Proof. Proof. Let $\langle p_\varepsilon \mid \varepsilon < \chi \rangle$ be a descending sequence of conditions for some limit ordinal $\chi < \mu$. Define p by setting $a_p = \bigcup \{\, a_{p_\varepsilon} \mid \varepsilon < \chi \,\}$, $\nu \leq_p \tau$ if there is some $\varepsilon < \chi$ such that $\nu \leq_{p_\varepsilon} \tau$ and $A_p = \bigcup \{\, A_{p_\varepsilon} \mid \varepsilon < \chi \,\}$.

First of all, I show that $p \in \mathbb{P}$. Suppose that η, ν, $\tau \in a_p$ and $\eta \in i_p(\nu, \tau)$. Let $\langle \xi_i \mid i \leq k \rangle$ and $\langle (\gamma_i, g_i) \mid i < k \rangle$ and $\langle \zeta_j \mid j \leq l \rangle$ and $\langle (\beta_j, f_j) \mid j < l \rangle$ be witnesses to the values of $c^p(\eta, \nu)$ and $c^p(\nu, \tau)$ respectively. Then there is some $\varepsilon < \chi$ such that $\{\, (\gamma_i, g_i) \mid i < k \,\} \cup \{\, (\beta_j, f_j) \mid j < l \,\} \subseteq A_{p_\varepsilon}$ and η, ν, $\tau \in a_{p_\varepsilon}$. First of all note, by the definition of i_{p_ε} and the fact that $a_{p_\varepsilon} \subseteq a_p$, that

$\eta \in i_{p_\varepsilon}(\nu, \tau)$. As p_ε is a condition this implies that $c^{p_\varepsilon}(\eta, \nu) \leq c^{p_\varepsilon}(\nu, \tau)$. But as $c^p(\eta, \nu) \leq c^{p_\varepsilon}(\eta, \nu)$ and $c^p(\nu, \tau) \leq c^{p_\varepsilon}(\nu, \tau)$, by Fact (1.13), one has that $\langle \xi_i \,|\, i \leq k \rangle$ and $\langle (\gamma_i, g_i) \,|\, i < k \rangle$ and $\langle \zeta_j \,|\, j \leq l \rangle$ and $\langle (\beta_j, f_j) \,|\, j < l \rangle$ are witnesses to the values of $c^{p_\varepsilon}(\eta, \nu)$ and $c^{p_\varepsilon}(\nu, \tau)$ respectively. Thus $c^p(\eta, \nu) = c^{p_\varepsilon}(\eta, \nu)$ and $c^p(\nu, \tau) = c^{p_\varepsilon}(\nu, \tau)$ and one has $c^p(\eta, \nu) \leq c^p(\nu, \tau)$ as required. Hence $p \in \mathbb{P}$.

In order to show that $p \leq p_\varepsilon$ for $\varepsilon < \chi$ one needs only to show that $i_p(\nu, \tau) = i_{p_\varepsilon}(\nu, \tau)$ for ν, $\tau \in a_{p_\varepsilon}$.

If $\eta \in i_p(\nu, \tau)$ there is some $\varepsilon' \in [\varepsilon, \chi)$ such that $\eta \in a_{p_{\varepsilon'}}$, and hence $\eta \in i_{p_{\varepsilon'}}(\nu, \tau)$ by the definition of $i_{p_{\varepsilon'}}$ (again). But $p_{\varepsilon'} \leq p_\varepsilon$, so $\eta \in a_{p_\varepsilon}(\nu, \tau)$.

Conversely, suppose towards a contradiction that $\eta \in i_{p_\varepsilon}(\nu, \tau)$ and there is some $\xi \in a_p$ with $\eta <_p \xi \leq_p \nu$, τ. Then there is some $\varepsilon' \in [\varepsilon, \chi)$ such that $\xi \in a_{p_{\varepsilon'}}$. Hence $\eta <_{p_{\varepsilon'}} \xi \leq_{p_{\varepsilon'}} \nu$, τ. But this contradicts the fact that $i_{p_{\varepsilon'}}(\nu, \tau) = i_{p_\varepsilon}(\nu, \tau)$, which is true because $p_{\varepsilon'} \leq p_\varepsilon$.

Lemma 16. For each $\xi < \kappa^+$ the collection $\{ p \in \mathbb{P} \,|\, \xi \in a_p \}$ is a dense (and open) subset of $\mathbb{P}$.

Proof. Let $\xi < \kappa^+$ and $p \in \mathbb{P}$. If $\xi \in a_p$ there is nothing to prove. Otherwise define q by $a_q = a_p \cup \{ \xi \}$, $\leq_q = \leq_p$, $A_q = A_p$. Then $q \in \mathbb{P}$ and $\xi \in a_q$.

Definition 17. Let G be $\mathbb{P}$-generic over V and define $B_\alpha = \{ \beta \leq \alpha \,|\, \exists p \in G \ \beta \leq_p \alpha \}$ for each $\alpha < \kappa^+$. Let $i(\alpha, \beta) = \{ \gamma \,|\, \exists p \in G \ \gamma \in i_p(\alpha, \beta) \}$ for $\alpha < \beta < \kappa^+$. Note that $\overline{\overline{i(\alpha, \beta)}} < \mu$ for all $\alpha < \beta < \kappa^+$.

Let $\mathcal{B} = \{ B_\alpha \setminus \bigcup \{ B_\beta \,|\, \beta \in b \} \,|\, \alpha < \kappa^+ \ \& \ b \in [\alpha]^{<\mu} \}$. Let τ be the topology on κ^+ generated by taking $\mathcal{B}$ as a sub-basis.

Lemma 18. $\mathcal{B}$ is a basis for a topology on κ^+, not just a sub-basis.

Proof. Let $\mathcal{O}_0 = B_\alpha \setminus \bigcup \{ B_\gamma \,|\, \gamma \in a \}$ and $\mathcal{O}_1 = B_\beta \setminus \bigcup \{ B_\varepsilon \,|\, \varepsilon \in b \}$ be sets in $\mathcal{B}$. Then $\mathcal{O}_0 \cap \mathcal{O}_1 = B_\alpha \cap B_\beta \cap (\kappa^+ \setminus \bigcup \{ B\gamma \,|\, \gamma \in a \cup b \})$.

Now $B_\alpha \cap B_\beta = \bigcup \{ B_\varepsilon \,|\, \varepsilon \in i(\alpha, \beta) \}$, so $\mathcal{O}_0 \cap \mathcal{O}_1 = \bigcup \{ B_\varepsilon \setminus \bigcup \{ B\gamma \,|\, \gamma \in a \cup b \} \,|\, \varepsilon \in i(\alpha, \beta) \}$. So in order to show that $\mathcal{O}_0 \cap \mathcal{O}_1$ is a union of sets in $\mathcal{B}$ it suffices to show that each $B_\varepsilon \setminus \bigcup \{ B_\gamma \,|\, \gamma \in a \cup b \}$ is in $\mathcal{B}$.

But $B_\varepsilon \setminus \bigcup\{ B_\gamma \mid \gamma \in a \cup b\} = B_\varepsilon \setminus \bigcup\{ B_\varepsilon \cap B_\gamma \mid \gamma \in a \cup b\}$ and each $B_\varepsilon \cap B_\gamma = \bigcup\{ B_\nu \mid \nu \in i\{\gamma, \varepsilon\}\}$ (recalling that $i\{\gamma, \varepsilon\}$ denotes $i(\gamma, \varepsilon$ if $\gamma \leq \varepsilon$, and $i(\varepsilon, \gamma)$ if instead $\varepsilon \leq \gamma$). Since each $i(\gamma, \varepsilon)$ has size less than μ, one has that $\{\nu \mid \exists\gamma \in a \cup b\ \nu \in i\{\gamma, \varepsilon\}\}$ is a union of fewer than μ sets each of size less than μ and so, as μ is regular, itself has size less than μ. Thus $B_\varepsilon \setminus \bigcup\{ B_\gamma \mid \gamma \in a \cup b\} \in \mathcal{B}$ as required and so $\mathcal{O}_1 \cap \mathcal{O}_2$ is a union of sets in $\mathcal{B}$.

Lemma 19. $\mathcal{B}$ is a clopen basis and so τ is 0-dimensional.

Proof. If $\mathcal{O} = B_\alpha \setminus \bigcup\{ B_\gamma \mid \gamma \in a\}$ then $\kappa^+ \setminus \mathcal{O} = (\kappa^+ \setminus B_\alpha) \cup \{ B_\gamma \mid \gamma \in a\}$. Clearly $B_\gamma \in \mathcal{B}$ for $\gamma \in a$. Also $\kappa^+ \setminus B_\alpha = \bigcup\{ B_\beta \setminus B_\alpha \mid \beta < \kappa^+\}$. But $B_\beta \setminus B_\alpha = B_\beta \setminus (B_\beta \cap B_\alpha) = B_\beta \setminus \bigcup\{ B_\nu \mid \nu \in i\{\alpha, \beta\}\}$, as in the proof of Lemma (18). Thus each $B_\beta \setminus B_\alpha$ is a union of sets in $\mathcal{B}$, and hence $\kappa^+ \setminus \mathcal{O}$ is also a union of sets in $\mathcal{B}$.

Lemma 20. τ is right-separated.

Proof. If $\alpha < \kappa^+$ then $\alpha \in B_\alpha$, but $B_\alpha \subseteq \alpha + 1$, so $\beta \notin B_\alpha$ for $\alpha < \beta < \kappa^+$. Of course, each $B_\alpha \in \mathcal{B}$.

Lemma 21. (κ^+, τ) is locally μ-Lindelöf.

Proof. Let $\alpha < \kappa^+$. $\alpha \in B_\alpha$, and B_α is closed (by Lemma (19)). So it suffices to show that each B_α is μ-Lindelöf. This is done by induction on κ^+.

Suppose that B_γ is μ-Lindelöf for each $\gamma < \alpha$. Let $\langle \mathcal{O}_i \mid i \in I\rangle$ be an open cover of B_α. So there is some $i \in I$ and some $B_\beta \setminus \bigcup\{ B_{\beta_j} \mid \beta_j \in b\} \subseteq \mathcal{O}_i$ such that $\alpha \in B_\beta \setminus \bigcup\{ B_{\beta_j} \mid \beta_j \in b\} \in \mathcal{B}$. So B_α has a subcover of size less than μ if and only if $B_\alpha \cap B_{\beta_j}$ has a subcover of size less than μ for each $j \in b$ (using the facts that $\overline{\overline{b}} < \mu$ and that μ is regular). But $B_\alpha \cap B_{\beta_j} = \bigcup\{ B_\gamma \mid \gamma \in i\{\alpha, \beta_j\}\}$ for each $j \in b$, and thus is the union of fewer than μ many sets of the form B_ε with $\varepsilon < \alpha$ since each $i\{\alpha, \beta_j\}$ has size less than μ.

By the inductive hypothesis each $B_\varepsilon, \subseteq B_\alpha \subseteq \bigcup\{\mathcal{O}_i \mid i \in I\}$, has a subcover of size less than μ, and hence $B_\alpha \cap B_{\beta_j}$ has the union of these subcovers as a subcover of size less than μ. Hence B_α is μ-Lindelöf in τ.

Lemma 22. Suppose $\mathcal{O} \in \mathcal{B}$. Then $\mathcal{O} \cap \mu \neq \emptyset$.

Proof. Let $p \in \mathbb{P}$ be such that $p \Vdash$ "$\mathcal{O} = B_\alpha \setminus \bigcup\{B_\gamma \mid \gamma \in a\} \in \mathcal{B}$" and suppose, without loss of generality, using the μ-closure of $\mathbb{P}$, that $\{\alpha\} \cup a \subseteq a_p$. Let $\beta \in \mu \setminus a_p$. Define q by $a_q = a_p \cup \{\beta\}$, $A_q = A_p$ and $\leq_q \restriction a_p \times a_p = \leq_p$ and for each $\varepsilon \in a_p$ set $\beta \leq_q \varepsilon$ if and only if $\alpha \leq_p \varepsilon$. It is clear that $q \in \mathbb{P}$ and $q \leq p$. And it is also clear that $q \Vdash \beta \in \mathcal{O} \cap \mu$.

Corollary 23. The separability degree of τ is μ.

Proof. Lemma (22) gives μ as an upper bound, but it is clear from the definition of $\mathbb{P}$ that given any smaller sized set and some condition there is a stronger condition that forces some set in $\mathcal{B}$ to miss it.

Proposition 24. τ has tightness μ.

Proof. Write $cl(Z)$ for the closure of Z in the topology generated by $\mathcal{B}$. I prove by induction that if $\gamma \in \mathrm{cl}(Z)$ for some $Z \subseteq \kappa^+$ then there is some $Y \in [Z]^\mu$ such that $\gamma \in \mathrm{cl}(Y)$. (*cf.* [M*1, Theorem 4.25]).

So fix $Z\kappa^+$, let $\gamma \in \mathrm{cl}(Z)$ and assume that for all $\varepsilon < \gamma$ there is First of all $Z \setminus B_\gamma \subseteq \kappa^+ \setminus B_\gamma$, so $\mathrm{cl}(Z \setminus B_\gamma) \subseteq \mathrm{cl}(\kappa^+ \setminus B_\gamma) = \kappa^+ \setminus B_\gamma$, since B_γ is open in τ. Thus $\mathrm{cl}(Z \setminus B_\gamma) \cap B_\gamma = \emptyset$.

This gives us that $\gamma \in \mathrm{cl}(Z \cap B_\gamma)$, because $\mathrm{cl}(Z) = \mathrm{cl}(Z \cap B_\gamma) \cup \mathrm{cl}(Z \setminus B_\gamma)$ since $\mathrm{cl}(.)$ is a closure operator.

Now B_γ is closed in τ by Lemma (19), so $\mathrm{cl}(X \cap B_\gamma) \subseteq B_\gamma$. I now proceed by a series of claims.

Claim 25. There is some $Y \subseteq Z$ such that $\gamma \in \mathrm{cl}(Y)$ and $\mathrm{cl}(Y) \cap \gamma$ has no maximal element.

Proof. Let $X_0 = Z \cap B_\gamma$ and inductively set δ_i to be maximal in $\mathrm{cl}(X_i) \cap \gamma$ and $X_{i+1} = X_i \setminus B_{\delta_i}$ for as long as possible. By the above observations applied to X_i in place of Z, $\mathrm{cl}(X_{i+1}) \subseteq (\kappa^+ \setminus B_{\delta_i})$ and $\mathrm{cl}(X_i \cap B_{\delta_i}) \subseteq B_{\delta_i}$, so that $\gamma \in \mathrm{cl}(X_{i+1}) \setminus \mathrm{cl}(X_i \cap B_{\delta_i})$ and $\delta_i \in \mathrm{cl}(X_i \cap B_{\delta_i}) \setminus \mathrm{cl}(X_{i+1})$ for each i for which δ_i is defined. As $X_{i+1} \subseteq X_i$ one has that $\delta_{i+1} \leq \delta_i$ for all i for which δ_{i+1} is defined. But $\delta_{i+1} \in \kappa^+ \setminus B_{\delta_i}$, so $\delta_{i+1} \neq \delta_i$ and, hence, $\delta_{i+1} < \delta_i$ for each i for which δ_{i+1} is defined. So let $k < \omega$ be least such that $\mathrm{cl}(X_k) \cap \gamma$ has no maximal element. Set $Y = X_k$. Then $Y \subseteq X_0 = Z$, $\gamma \in \mathrm{cl}(Y)$ and $\mathrm{cl}(Y) \cap \gamma$ has no maximal element.

Claim 26. There is some $Y' \in [\mathrm{cl}(Y) \cap \gamma]^\mu$ such that $\gamma \in \mathrm{cl}(Y')$.

Proof. Let W be cofinal in $\mathrm{cl}(Y) \cap \gamma$. Choose $Y' \in [W]^\mu$ such that $Y^* = \mu \cap \bigcup\{ B_\delta \mid \delta \in W \} \subseteq \bigcup\{ B_\delta \mid \delta \in Y' \}$. Then $\mathrm{cl}(Y') \subseteq \mathrm{cl}(\mathrm{cl}(Y) \cap \gamma) \subseteq \mathrm{cl}(\mathrm{cl}(Y)) = \mathrm{cl}(Y) \subseteq \gamma + 1$, so $\mathrm{cl}(Y')$ is μ-Lindelöf in τ and there are $\{ \alpha_i \mid i < \chi \}$ for some $\chi < \mu$ such that $Y' \subseteq \bigcup\{ B_{\alpha_i} \mid i < \chi \}$. Consequently, if $\varepsilon \in Y'$ there is some $i < \chi$ with $\varepsilon \in B_{\alpha_i}$ and hence $B_\varepsilon \subseteq B_{\alpha_i}$. Hence $\bigcup\{ B_\delta \mid \delta \in Y' \} \subseteq \bigcup\{ B_{\alpha_i} \mid i < \chi \}$ and thus $Z \subseteq \bigcup\{ B_{\alpha_i} \mid i < \chi \}$.

However, if $\alpha_i < \gamma$ for all $i < \chi$ then there is some $\delta \in \mathrm{cl}(Y) \cap \gamma$ such that $\alpha_i < \delta$ for all $i < \chi$. But then $Z \cap (B_\delta \setminus \bigcup\{ B_{\alpha_i} \mid i < \chi \}) = \mu \cap (B_\delta \setminus \bigcup\{ B_{\alpha_i} \mid i < \chi \}) \cap \bigcup\{ B_\delta \mid \delta \in W \}$. Now $B_\delta \setminus \bigcup\{ B_{\alpha_i} \mid i < \chi \} \neq \emptyset$ as $\alpha_i < \delta$ for each $i < k$ and hence $\delta \in B_\delta \setminus \bigcup\{ B_{\alpha_i} \mid i < \chi \}$. So $\mu \cap (B_\delta \bigcup\{ B_{\alpha_i} \mid i < \chi \})$ by Lemma (22). But any element of this set is an element of B_δ and hence of $\bigcup\{ B_\delta \mid \delta \in W \}$. So one has $Z \cap (B_\delta \setminus \bigcup\{ B_{\alpha_i} \mid i < \chi \}) \neq \emptyset$, a contradiction.

Hence there is some i such that $\alpha_i = \gamma$, and thus $\gamma \in \mathrm{cl}(Y')$.

Claim 27. There is some $Y'' \in [Y']^\mu$ such that $\gamma \in \mathrm{cl}(Y'')$.

Proof. If $\delta \in Y'$ then $\delta \in \mathrm{cl}(Z) \cap \gamma$ and so by the inductive hypothesis there is some $W_\delta \in [Z]^\mu$ such that $\delta \in \mathrm{cl}(W_\delta)$. Set $Y'' = \bigcup\{ W_\delta \mid \delta \in Y' \}$. Then $\overline{\overline{Y''}} \leq \mu$, and $Y \subseteq \mathrm{cl}(Y'')$, so one has that $\gamma \in \mathrm{cl}(Y') \subseteq \mathrm{cl}(\mathrm{cl}(Y'')) = \mathrm{cl}(Y'')$.

By Lemmas (18) to (23) and Proposition (24), (κ^+, τ) has the topological properties described in Theorem (1) and cardinals are preserved in the generic extension by Propositions (5), (13) and (15).

Comment. As remarked after Definition (2) it seems unlikely that one can force to add a 0-dimensional, locally compact, countably tight, scattered space of size κ^+ by κ-$\mathbb{M}$-proper forcing when μ is uncountable. It is hard to see how to amalgamate two conditions whose working parts are infinite, albeit of size less than μ, and keep the size of the ranges of an analogue of i_r finite. Of course the consistency of there being such spaces is a major, and radically open, problem.

§4. Chains in $\mathcal{P}(\kappa)$ mod $<\mu$.

Theorem. If there is a stationary $(\kappa,1)$-simplified morass and κ is the successor of a regular cardinal μ with $2^{<\mu} \leq \kappa$ then there is a cardinal preserving forcing which adds a chain of length κ^+ in $\mathcal{P}(\kappa)$ mod $<\mu$.

In this section I show how to use κ-$\mathbb{M}$-proper forcing to add a chain of length κ^+ in $\mathcal{P}(\kappa)$ mod $<\mu$, that is a sequence $\langle X_\nu \mid \nu < \kappa^+ \rangle$ of subsets of κ such that $\overline{\overline{X_\nu \setminus X_\tau}} < \mu$ and $\overline{\overline{X_\tau \setminus X_\nu}} = \kappa$ for each $\nu < \tau < \kappa^+$. One has that if the Chang conjecture $(\kappa^{++}, \kappa^+) \longrightarrow (\kappa^+, \kappa)$ holds then there is no such chain — Koszmider's proof ([K98, Fact 1]) that this is so when $\kappa = \omega_1$ goes through for arbitrary successor κ. The forcing is the obvious generalisation of Koszmider's forcing from [K98] with sets of maps from the simplified morass of size less than μ as side conditions.

Definition 1. Let the forcing $\mathbb{P}$ consist of conditions $p = (a_p, b_p, f_p, A_p)$ where $a_p \in [\kappa^+]^{<\mu}$, $b_p \in [\kappa]^{<\mu}$, $f_p : a_p \times b_p \longrightarrow 2$ and $A_p \in [\mathcal{F}]^{<\mu}$ and, writing c^p for c_{A_p},

$$(\bullet) \qquad \forall \nu, \tau \in a_p \ \forall \beta \in b_p \ \big((\nu < \tau \ \ \& \ \ c^p(\nu,\tau) \leq \beta) \longrightarrow (f_p(\nu,\beta) = 1 \longrightarrow f_p(\tau,\beta) = 1)\big).$$

$q \leq p$ if $a_p \subseteq a_q$, $b_p \subseteq b_q$, $f_p = f_q \restriction a_p \times b_p$, $A_p \subseteq A_q$ and

$$(\circ) \qquad \forall \nu, \tau \in a_p \ \forall \beta \in b_q \setminus b_p \ \big(\nu < \tau \longrightarrow (f_q(\nu,\beta) = 1 \longrightarrow f_q(\tau,\beta) = 1)\big).$$

The idea behind the forcing is that if G is $\mathbb{P}$-generic over the ground model then one gets the desired chain in $\mathcal{P}(\kappa)$ mod $<\mu$ by setting $X_\nu = \{\beta < \kappa \mid \exists p \in G \ f_p(\nu,\beta) = 1\}$ for $\nu < \kappa^+$. The following two lemmas say that this will happen.

Lemma 2. $\{p \in \mathbb{P} \mid \beta \in b_p\}$ is dense and open for each $\beta < \kappa$.

Proof. If $\beta \notin b_p$ then define q by setting $a_q = a_p$, $A_q = A_p$, $b_q = b_p \cup \{\beta\}$, $f_q \restriction a_p \times b_p = f_p$ and $f_q(\nu,\beta) = 0$ for all $\nu \in a_p$. Then clearly $q \in \mathbb{P}$, $q \leq p$ and $q \in \{p \in \mathbb{P} \mid \beta \in b_p\}$.

Lemma 3. $\{p \in \mathbb{P} \mid \tau \in a_p\}$ is dense and open for each $\tau < \kappa^+$.

Proof. If $\tau \notin a_p$ then define q by setting $a_q = a_p \cup \{\tau\}$, $b_q = b_p$, $A_q = A_p$, $f_q \restriction a_p \times b_p = f_p$, and by setting, for each $\beta \in b_p$, $f_q(\tau,\beta) = 1$ if there is some $\nu \in a_p \cap \tau$ such that $c^q(\nu,\tau) \leq \beta$ and

$f_p(\nu,\beta) = 1$, and $f_q(\tau,\beta) = 0$ otherwise. If $\beta \in b_p$, and $f_q(\tau,\beta) = 1$ then by the definition just given of f_q there is some $\nu < \tau$ such $c^q(\nu,\tau) \le \beta$ and $f_p(\nu,\beta) = 1$. If one also has $\xi \in a_p \setminus \tau$ and $c^q(\tau,\xi) \le \beta$ then $c^q(\nu,\xi) \le \beta$ by the subadditivity of c^q. As $c^q = c^p$, the condition ($\bullet$) for p applied to ν and ξ ensures that $f_p(\xi,\beta) = 1$. This shows that the condition ($\bullet$) also holds for q, so that $q \in \mathbb{P}$. It is clear that $q \le p$ (the condition ($\circ$) holds vacuously since $b_q \setminus b_p = \emptyset$).

It remains to show that $\mathbb{P}$ preserves cardinals.

Lemma 4. $\mathbb{P}$ is μ-closed.

Proof. Let $\langle p_i \mid i < \chi \rangle$ be a descending sequence of conditions for some limit ordinal $\chi < \mu$. Define p by setting $a_p = \bigcup\{ a_{p_i} \mid i < \chi \}$, $b_p = \bigcup\{ b_{p_i} \mid i < \chi \}$, $f_p = \bigcup\{ f_{p_i} \mid i < \chi \}$, and $A_p = \bigcup\{ A_{p_i} \mid i < \chi \}$.

First of all I show that ($\bullet$) holds for p and hence that $p \in \mathbb{P}$. Suppose that ν, $\tau \in a_p$, $\beta \in b_p$, $c^p(\nu,\tau) \le \beta$ and $f_p(\nu,\beta) = 1$. Let $c^p(\nu,\tau) = \alpha$ and let $\langle \xi_j \mid j \le k \rangle$ and $\langle (\gamma_j, g_j) \mid j < k \rangle$ witness this. Then there is some $i < \chi$ such that ν, $\tau \in a_{p_i}$, $\beta \in b_{p_i}$ and $(\gamma_j, g_j) \in A_{p_i}$ for each $j < k$. So $f_{p_i}(\nu,\beta) = 1$ and $\langle \xi_j \mid j \le k \rangle$ and $\langle (\gamma_j, g_j) \mid j < k \rangle$ witness that $c^{p_i}(\nu,\tau) \le \alpha$. (As $A_{p_i} \subseteq A_p$ this in fact gives that $c^{p_i}(\nu,\tau) = c^p(\nu,\tau)$ by Fact (1.13).) By ($\bullet$) for p_i one has that $f_{p_i}(\tau,\beta) = 1$, and hence $f_p(\tau,\beta) = 1$.

Next I show that $p \le p_i$ for all $i < \chi$ by checking that ($\circ$) holds between p and each p_i. So let $i < \chi$ with ν, $\tau \in a_{p_i}$, $\nu < \tau$, $\beta \in b_p \setminus b_{p_i}$ and $f_p(\nu,\beta) = 1$. Then there is some $i' \in [i,\chi)$ such that $\beta \in p_{i'}$, whence $f_{p_{i'}}(\nu,\beta) = 1$ since $f_p \restriction a_{p_{i'}} \times b_{p_{i'}} = f_{p_{i'}}$. By ($\circ$) for $p_{i'}$ and p_i one has that $f_{p_{i'}}(\tau,\beta) = 1$ and hence that $f_p(\tau,\beta) = 1$, as required.

Lemma 5. $\mathbb{P}$ has the κ^+-chain condition.

Proof. Let $\{ p_i \mid i < \kappa^+ \}$ be a subset of $\mathbb{P}$ of size κ^+. For each $p' \in \{ p'_i \mid i < \kappa^+ \}$, by Fact (1.5), let $(\gamma, g) \in \mathcal{F}$ be such that $a_{p'} \subseteq \mathrm{rge}(g)$, $\mathrm{ssup}(b_{p'}) < \gamma$ and for all $(\beta, f) \in A_{p'}$ there is some $f' \in \mathcal{F}_{\beta\gamma}$ with $f = g \cdot f'$. Define a condition p by letting $a_p = a_{p'}$, $b_p = b_{p'}$, $f_p = f_{p'}$ and $A_p = A_{p'} \cup \{ (\gamma, g) \}$. Now it is clear that $c^{p'}(\nu,\tau) \le \beta$ if and only if $c^p(\nu,\tau) \le \beta$ for any pair ν, $\tau \in a_p$ and $\beta \in b_p$ since any sequence of maps witnessing the latter cannot contain (γ, g) (as $\beta < \gamma$) and, hence, witnesses the former. Hence p satisfies ($\bullet$) and so is a condition. As ($\circ$) holds vacuously and all of the other conditions for being an extension hold by definition one also has that $p \le p'$. If one can find i,

$j < \kappa^+$ such that p_i and p_j are compatible then, clearly, one will have that p'_i and p'_j are compatible as well.

By thinning using the Δ-system lemma if necessary one may as well assume that there is some $b \in [\kappa]^{<\mu}$ such that $b = b_{p_i}$ for all $i < \kappa^+$ since $\kappa^{<\mu} = \kappa$. One may as well also assume there is some $a \in [\kappa^+]^{<\mu}$ such that a is an initial segment of each a_{p_i}, that if $i < j < \kappa^+$ then $\max(a_{p_i}) < \min(a_{p_j} \setminus a)$, and there is an order preserving isomorphism from a_{p_i} to a_{p_j} which extends to an isomorphism from f_{p_i} to f_{p_j}. Moreover, one may as well assume that there is some $\alpha < \kappa$, some $A \in [\bigcup\{\{\beta\} \times \mathcal{F}_{\beta\alpha} \mid \beta < \alpha\}]^{<\mu}$ and a map $F_i \in \mathcal{F}_{\alpha\kappa}$ for each $\nu < \kappa^+$ such that $A_{p_i} = \{(\beta, F_\nu \cdot h) \mid (\beta, h) \in A\} \cup \{(\alpha, F_i)\}$ and such that $F_i \cdot F_j^{-1} \restriction a_{p_j}$ is the order preserving isomorphism from (a_{p_j}, f_{p_j}) to (a_{p_i}, f_{p_j}). This is possible using the directedness of $\mathcal{F}$ and then applying the Δ-system lemma. Clearly for each $i < \kappa^+$ and $\xi < \zeta < \kappa^+$ one has that $c^{p_i}(\xi, \zeta) \leq \alpha$ if and only if $c^{p_i}(\xi, \zeta) < \kappa$.

Let $i < j < \kappa^+$ and set $a_r = a_{p_i} \cup a_{p_j}$, $b_r = b$, $f_r \restriction a_{p_i} \times b = f_{p_i}$ and $f_r \restriction a_{p_j} \times b = f_{p_j}$, and $A_r = A_{p_i} \cup A_{p_j}$.

It is also clear that if $\xi < \zeta < \kappa^+$ then $c^r(\xi, \zeta) \leq \alpha$ if and only if $c^r(\xi, \zeta) < \kappa$. By Fact (1.19), one has that if $\zeta \in \mathrm{rge}(F_i)$ then $c^r(\xi, \zeta) = c^{p_i}(\xi, \zeta)$ and if $\zeta \in \mathrm{rge}(F_j)$ then $c^r(\xi, \zeta) = c^{p_j}(\xi, \zeta)$.

Claim 6. If $\xi \in a_{p_i} \setminus a_{p_j}$ then $\xi \notin \mathrm{rge}(F_j)$.

Proof. If $\xi \in \mathrm{rge}(F_j)$ then $\xi \in \mathrm{rge}(F_j) \cap \mathrm{rge}(F_i)$ since $F_j \cdot F_i^{-1} \restriction a_{p_i}$ is the order preserving isomorphism from a_{p_i} to a_{p_j}. But then one has that $F_j(\xi_\alpha) = \xi = F_i(\xi_\alpha)$ by Fact (1.3), so $F_j \cdot F_i^{-1}(\xi) = \xi$ and hence $\xi \in a_{p_i} \cap a_{p_j}$, a contradiction.

It is now immediate that if $\xi \in a_{p_i} \setminus a_{p_j}$ and $\zeta \in a_{p_j} \setminus a_{p_i}$ then $\xi \notin \mathrm{rge}(F_j)$, by Claim (6), so that $c^{p_i}(\xi, \zeta) = \kappa$, and thus $c^r(\xi, \zeta) = \kappa$ from Fact (1.19). Hence for such ξ, ζ one has that $\beta < c^r(\xi, \zeta)$ for all $\beta \in b$. Hence ($\bullet$) is vacuously true for all such pairs ξ, ζ.

By Claim (6) as well, if ξ_0, $\xi_1 \in a_{p_i}$ then $c^r(\xi_0, \xi_1) = c^{p_i}(\xi_0, \xi_1)$. And so if $\xi_0 < \xi_1$, $c(\xi_0, \xi_1) \leq \beta$ and $f_r(\xi_0, \beta) = 1$ for some $\beta \in b$ then (by ($\bullet$) for p_i) one also has that $f_r(\xi_1, \beta) = 1$. Similarly, if ζ_0, $\zeta_1 \in a_{p_j}$, $c(\xi_0, \xi_1) \leq \beta$ and $f_r(\zeta_0, \beta) = 1$ for some $\beta \in b$ then (by ($\bullet$) for p_j) one also has that $f_r(\zeta_1, \beta) = 1$.

Hence $r \in \mathbb{P}$, and thus $r \leq_{\mathbb{P}} p_i$, p_j (since ($\circ$) holds vacuously). So $\mathbb{P}$ does have the κ^+-cc.

The last two lemmas show that cardinals below κ and above κ, respectively, are preserved. So in order to complete the proof it suffices to show that $\mathbb{P}$ is κ-$\mathbb{M}$-proper and hence that κ itself is preserved.

Proposition 7. $\mathbb{P}$ is κ-$\mathbb{M}$-proper.

Proof. The proof strategy will be to work through the framework of §2 checking that one can actually perform each of the four steps enumerated there for $\mathbb{P}$.

So suppose that $p \in \mathbb{P}$ and that $\mathcal{N}$ is as in the general framework but for this specific $\mathbb{P}$. Obviously, a_p is the realm of p and A_p is the side condition part.

Step A. Checking that $(a_p, b_p, f_p, A_p \cup \{(\delta, F)\})$ is a condition and is stronger than p.

Proof. This is just as in the argument of the first paragraph of the proof of the κ^+-chain condition (Lemma (5)). Let $p' = (a_p, b_p, f_p, A_p \cup \{(\delta, F)\})$. Clearly p' is a condition if it satisfies ($\bullet$) and is stronger than p if it is a condition because ($\circ$) will hold vacuously. But if ν, $\tau \in a_p$, $\beta \in b_p$ and $c^{p'}(\nu, \tau) \leq \beta$ then any maps in a witnessing sequence to this cannot include (δ, F) because $p \in \mathcal{N}$ and has size less than μ, so $b_p \subseteq \mathcal{N} \cap \kappa = \delta$. Hence any maps from such a witnessing sequence witness that $c^p(\nu, \tau) \leq \beta$ as well. Thus if $p(\nu, \beta) = 1$ one also has that $p(\tau, \beta) = 1$ by ($\bullet$) for p.

Now let $\mathcal{D}$ be a dense, open subset of $\mathbb{P}$ and $q \in \mathcal{D}$ with $q \leq p'$.

Step B. One may assume without loss of generality that $c^q(\nu, \tau) < \kappa$ for all ν, $\tau \in a_q$

Proof. The argument is almost exactly the same argument as in the first paragraph of the proof of the κ^+-chain condition (Lemma (5)) and the argument used for Step (A). Explicitly, suppose one has a q without this property. Then for each $\nu \in a_q$ choose a map $(\gamma_\nu, g_\nu) \in \mathcal{F}$ such that $\nu \in \mathrm{rge}(g_\nu)$. Now apply Fact (1.5) to this collection of maps and select some $(\gamma, g) \in \mathcal{F}$ through which they all factor and with γ greater than $\mathrm{ssup}(\{\alpha \mid \exists(\alpha, f) \in A_q\} \cup b_p)$. Add this single map (γ, g) to A^q. As in the argument for Step (A) the resulting object is still a condition since the new map cannot play a rôle in witnessing that $c(\nu, \tau) \leq \beta$ for any $\beta \in b_q$ and ν, $\tau \in a_q$.

Next let $q \restriction \mathcal{N} = (a_q \cap \mathcal{N}, b_q \cap \mathcal{N}, f_q \cap \mathcal{N}, A_q \cap \mathcal{N})$. Note that $f_q \cap \mathcal{N} = f_q \restriction (a_q \cap \mathcal{N} \times b_q \cap \mathcal{N})$, since the range of f_q is (a subset of) $\{0, 1\}$. This makes the proof that $q \restriction \mathcal{N}$ particularly simple.

Step C. Show that $q \restriction \mathcal{N} \in \mathbb{P} \cap \mathcal{N}$ and that $q \leq q \restriction \mathcal{N}$.

Proof. Each of the four components is a set of size less than μ which is an element of $\mathcal{N}$ and so is itself an element of $\mathcal{N}$ since the latter is closed under sequences of length less than μ. Thus $q \restriction \mathcal{N} \in \mathcal{N}$.

Now I prove that $q \restriction \mathcal{N} \in \mathbb{P}$ by showing that $(\bullet)$ holds. Let $\nu,\ \tau \in a_q \cap \mathcal{N}$, $\beta \in b_q \cap \mathcal{N} (= b_q \cap \delta)$, $\nu < \tau$, $c^{q \restriction \mathcal{N}}(\nu, \tau) \leq \beta$ and $f_q(\nu, \beta) = 1$. By Fact (1.13) one has that $c^q(\nu, \tau) \leq c^{q \restriction \mathcal{N}}(\nu, \tau) \leq \beta$, so one has that $f_q(\tau, \beta) = 1$ by $(\bullet)$ for q. Thus $q \restriction \mathcal{N}$ does satisfy $(\bullet)$ and so $q \restriction \mathcal{N} \in \mathbb{P}$.

Finally I prove that $q \leq q \restriction \mathcal{N}$ by demonstrating that $(\circ)$ holds. Let $\nu,\ \tau \in a_q \cap \mathcal{N}$, again, with $\nu < \tau$, let $\beta \in b_q \setminus \mathcal{N}$ and suppose that $f_q(\nu, \beta) = 1$. As $\nu,\ \tau \in \mathcal{N} = \mathrm{rge}(F)$, $(\delta, F) \in A_q$ and $\beta \notin \mathcal{N}$ one has that $c^q(\nu, \tau) \leq \delta \leq \beta$. Hence, by $(\bullet)$ for q, one has that $f_q(\tau, \beta) = 1$. Thus $(\circ)$ does hold between q and $q \restriction \mathcal{N}$ and one does have that $q \leq q \restriction \mathcal{N}$.

Now obtain s as in §2 using b_q as the arbitrary set e_q in the section of Notation in §2.

Step D. Define some r with realm $a_s \cup a_q$ and side condition part $A_s \cup A_q$. Show that $r \in \mathbb{P}$ and that $r \leq s,\ q$.

Proof. Write a for $a_s \cap a_q$, a_0 for $a_s \setminus a_q$, a_1 for $a_q \setminus a_s$, b for $b_s \cap b_q$, b_0 for $b_s \setminus b_q$, b_1 for $b_q \setminus b_s$.

Define r as follows. Let $a_r = a_s \cup a_q$, $b_r = b_s \cup b_q$ and $A_r = A_s \cup A_q$. If $(\tau, \beta) \in a_s \times b_s$ let $f_r(\tau, \beta) = f_s(\tau, \beta)$ and if $(\tau, \beta) \in a_q \times b_q$ let $f_r(\tau, \beta) = f_q(\tau, \beta)$. It remains to define $f_r \restriction a_0 \times b_1 \cup a_1 \times b_0$. This is done separately for each β and by induction on τ.

If $\beta \in b_1$ and $\tau \in a_0$ let $f_r(\tau, \beta) = 1$ if there is some $\nu \in a_r \cap \tau$ such that $c^r(\nu, \tau) \leq \beta$ and $f_r(\nu, \beta) = 1$ or if there is some $\nu \in a_0 \cap \tau$ such that $f_r(\nu, \beta) = 1$. Otherwise set $f_r(\nu, \tau) = 0$.

Similarly, if $\beta \in b_0$ and $\tau \in a_1$ let $f_r(\tau, \beta) = 1$ if there is some $\nu \in a_r \cap \tau$ such that $c^r(\nu, \tau) \leq \beta$ and $f_r(\nu, \beta) = 1$ or there is some $\nu \in a_1 \cap \tau$ such that $f_r(\nu, \beta) = 1$. Otherwise set $f_r(\nu, \tau) = 0$.

Observe that if $\tau \in a_1$ and $f_r(\tau, \beta) = 1$ because there is some $\nu \in a_1$ such that the inductive process already set $f_r(\nu,\beta)=1$ then there is some such ν (obtainable by taking the least such ν) for which there is some $\zeta \in a_s$ with $c^r(\zeta,\nu) < \beta$ and $f_s(\zeta,\beta)=1$. A similar observation is true for a_0. These observations are used repeatedly below, without explicit comment.

I show, first of all, that r is a condition by proving that it satisfies ($\bullet$). Suppose that ν, $\tau \in a_r$ with $\nu < \tau$, $\beta \in b_r$, $c^r(\nu,\tau) \leq \beta$ and $f_r(\nu,\beta)=1$. If $\tau \in a_1$ and $\beta \in b_0$ or $\tau \in a_0$ and $\beta \in b_1$ then $f_r(\tau,\beta)=1$ by the definition of f_r just made.

Case A. $\tau \in a_q$ and $\beta \in b_q$. Then there is some $\xi \in a_q \cap \tau$ such that $f_r(\xi,\beta)=1$. For if $\nu \in a_q$ then take $\xi=\nu$. Otherwise, if $\nu \in a_0$ then there is some $\zeta \leq \nu$ such that $\zeta \in a_0$ and some $\xi \in a_q \cap \zeta$ such that $c^r(\xi,\zeta) \leq \beta$ and $f_r(\xi,\beta)=1$. As ν, $\zeta \in a_s$ one has that $c^r(\zeta,\nu) < \delta$ (by [M*2,4.12]), so $c^r(\xi,\tau) \leq \beta$ by the subadditivity of c^r (Fact (1.20)).

Subcase A.i. $\beta \in b$. Then $\beta < \alpha^*$ (by clause (iii) of the definition of ϕ for s) and one has that $c^r(\xi,\tau) = c^q(\xi,\tau)$ by [M*2, 4.15]. Thus $f_q(\tau,\beta)=1$ by ($\bullet$) for q and so $f_r(\tau,\beta)=1$.

Subcase A.ii. $\beta \in b_1$. One has that $\delta \leq \beta$ (by clause (iii.5) of the definition of ϕ from the Notation of §2). Hence, by [M*2, 4.16], one has that $c^q(\xi,\tau) \leq \beta$ So one has that $f_q(\tau,\beta)=1$ by ($\bullet$) for q and hence $f_r(\xi,\tau)=1$.

Case B. $\tau \in a_s$ and $\beta \in b_s$. One cannot have that $\nu \in a_1$ because then one would have $\beta < \delta \leq c^r(\xi,\tau)$, by [M*2, 4.17]. So suppose that $\nu \in a_s$. Then $c^s(\nu,\tau) = c^r(\nu,\tau) \leq \beta < \delta$ by [M*2, 4.12], and so $f_s(\tau,\beta)=1$ by ($\bullet$) for s, whence $f_r(\tau,\beta)=1$ again.

Now I show that $r \leq q$. Let ν, $\tau \in a_q$, $\nu < \tau$, $\beta \in b_0$ and $f_r(\nu,\beta)=1$. If $\tau \in a_0$ then $f_r(\tau,\beta)=1$ by the definition of $f_r(.\,,\beta)$. So suppose that $\tau \in a_s \cap a_q$.

Case A. $\nu \in a_s \cap a_q$. Then ν, $\tau \in \mathrm{rge}(h^*)$. This shows that $c^r(\nu,\tau) \leq \beta < \delta$ for otherwise one has $\beta < c^r(\nu,\tau) \leq \alpha^*$, but $b_s \setminus b_q \cap \alpha^* = \emptyset$, a contradiction. Now, $c^s(\nu,\tau) = c^r(\nu,\tau) \leq \beta$, by [M*2, 4.12], so $f_s(\tau,\beta)=1$ by ($\bullet$) for s, and hence $f_r(\tau,\beta)=1$.

Case B. $\nu \in a_1$. Let $\xi \in a_1 \cap \nu$ and $\zeta \in a_s \cap \xi$ be such that $c^r(\zeta,\xi) \leq \beta$ and $f_r(\zeta,\beta)=1$. If $\zeta \in a_s \cap a_q$ then Case (A) deals with the situation. If $\zeta \in a_0$ then $c^s(\zeta,\tau) = c^r(\zeta,\tau) \leq c^r(\zeta,\xi)$, by [M*2, 4.12] and [M*2, 4.22] respectively. So $c^s(\zeta,\tau) \leq \beta$, and $f_s(\tau,\beta)=1$ by ($\bullet$) for s. Hence $f_r(\tau,\beta)=1$.

Finally, I show that $r \leq s$. Let $\nu, \tau \in a_s$, $\nu < \tau$, $\beta \in b_1$ and $f_r(\nu, \beta) = 1$. If $\tau \in a_1$ then $f_r(\tau, \beta) = 1$ by the definition of $f_r(.\,, \beta)$. So suppose that $\tau \in a_s \cap a_q$.

Case A. $\nu \in a_s \cap a_q$. Then $c^q(\nu, \tau) \leq \delta \leq \beta$ since $\nu, \tau \in \mathrm{rge}(F)$ so one has that has that $f_q(\tau, \beta) = 1$ by ($\bullet$) for q, whence $f_r(\tau, \beta) = 1$.

Case B. $\nu \in a_0$. Let $\zeta \in a_0 \cap \nu$ and $\xi \in a_s \cap \zeta$ be such that $c^r(\xi, \zeta) \leq \beta$ and $f_r(\xi, 1) = 1$. If $\xi \in a_s \cap a_q$ then one is back in Case (A) again and so finished. If $\xi \in a_1$ then $\delta \leq c^r(\xi, \zeta) = c^r(\xi, \nu) = c^q(\xi, \nu)$ by [M*2, 4.17], the subadditivity of c^r and the fact that $c^r(\zeta, \nu) < \delta$ by [M*2, 4.12]. But, again by [M*2, 4.12], $c^r(\nu, \tau) < \delta$. So $\delta \leq c^r(\xi, \nu) = c^r(\xi, \tau) = c^q(\xi, \tau)$, by the subadditivity of c^r once more. Thus $c^q(\xi, \tau) \leq \beta$, and $f_r(\tau, \beta) = 1$ by ($\bullet$) for q. Hence $f_r(\xi, \tau) = 1$.

This concludes the proof that $\mathbb{P}$ is κ-$\mathbb{M}$-proper and so the proof of this section's Theorem.

Comment. I do not know whether there is a κ-chain condition, μ-closed forcing to add such a chain, and in particular whether using the coupling function from a $(\kappa, 1)$-simplified morass one can generalise Koszmider's proof for $\kappa = \omega_1$ directly.

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