Laver Indestrucibility and Strong Compactness

Arthur W. Apter
Department of Mathematics
Baruch College of CUNY
New York NY 10010
USA

Set Theory, Analysis, and their Neighbours (STAATN) Meeting
January 7, 1999
London, England

We take as our convention that sc means supercompact and strc means strongly compact.

In the mid '70s, Richard Laver proved the following:

Theorem 1 *(Laver, Israel Journal of Mathematics, 1978):*

Let $V \models$ "ZFC + κ is sc". There is then a p.o. $\mathbb{P} \in V$, $|\mathbb{P}| = \kappa$, s.t. $V^{\mathbb{P}} \models$ "κ is sc and κ's scness is indestructible under κ-directed closed forcing". This means that if $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is κ-directed closed", then $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ is sc".

Remarks:

1. A sc card. κ as above is called *Laver indestructible.*

2. Laver indestructibility has been a fundamental tool in large cardinals and forcing since its discovery. Its first application was by Magidor, who used it in his construction, starting from a sc card. with a huge card. above it, of a model in which GCH holds below $\aleph_\omega$ and $2^{\aleph_\omega} = \aleph_{\omega+2}$.

We can ask the following question:

If κ is a non-sc strc card., what sort of indestructibility properties can κ exhibit?

There are two ways of approaching the above question. We can either use properties of scness, or we can see what can be proven using an arbitrary strc card.

Some results using the first of these approaches are as follows:

Theorem 2 *(A., JSL '98):*

Let $V \models$ "ZFC + κ is a measurable limit of sc cards. + $2^\kappa = \kappa^+$". There is then a p.o. $\mathbb{P} \in V$, $|\mathbb{P}| = \kappa$ s.t. $V^{\mathbb{P}} \models$ "κ is a measurable limit of sc cards." Further, if $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is a κ-directed closed p.o. which doesn't add subsets to κ", then $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ is a meas. limit of sc cards."

Remarks:

1. Menas showed that any measurable limit of strc cards. is strc. He also showed that many measurable limits of strc cards. aren't strc.

2. Theorem 2 thus shows that certain non-sc strc cards. can exhibit a weak form of indestructibility. Examples of p.o.s $\mathbb{Q}$ as in Theorem 2 include any κ^+-directed closed p.o., the p.o. adding a branch to a normal Souslin tree on κ^+, the p.o. adding a κ-club to a κ-stationary subset of κ^+, and the p.o. which adds to any regular card. $\lambda > \kappa$ a non-reflecting stationary set of ordinals of cofinality κ.

Theorem 3 *(A. & Gitik, JSL '99):*

Let $V \models$ "ZFC + κ is sc". There is then a p.o. $\mathbb{P} \in V$, $|\mathbb{P}| = \kappa$ s.t. $V^{\mathbb{P}} \models$ "κ is both the least measurable and least strc card." Further, if $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is κ-directed closed", then $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ is both the least measurable and least strc card."

Remark: Theorem 3 thus shows, since the least measurable card. can't be sc, that it is consistent for a non-sc strc card. to be fully Laver indestructible.

Theorem 4 *(Hamkins, Fall '97):*

Let $V \models$ "ZFC + κ is a sc limit of sc cards". There is then a p.o. $\mathbb{P} \in V$, $|\mathbb{P}| = \kappa$ s.t. $V^{\mathbb{P}} \models$ "κ is the least measurable limit of sc cards." Further, if $\Vdash_{\mathbb{P}}$ "$\dot{\mathbb{Q}}$ is κ-directed closed", then $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ "κ is the least measurable limit of sc cards."

Remark: Menas' work shows the least measurable limit of sc (or strc) cards. isn't sc. Thus, Theorem 4 provides a model in which this particular non-sc strc card. is fully Laver indestructible, whereas Theorem 2 provides a model in which this particular non-sc strc. card. has a weak form of indestructibility.

Hamkins has studied what can be proven using the second of the above approaches, i.e., what indestructibility properties can be established using an arbitrary non-sc strc card. He has shown the following:

Theorem 5 *(Hamkins, to appear in the APAL):*

Let $V \models$ "ZFC + κ is strc". There is then a p.o. $\mathbb{P} \in V$, $|\mathbb{P}| = \kappa$ s.t. $V^{\mathbb{P}} \models$ "κ is strc and κ's strcness is indestructible under forcing with the p.o. adding one Cohen subset to κ, certain p.o.s which add clubs to κ, and certain p.o.s which add "long Prikry sequences" to κ".

Note that although Laver's forcing is iterable so that we can fairly easily make any class $\mathcal{K}$ of scs simultaneously indestructible, the forcing of Theorem 3 (which makes the least strc be both the least measurable and be fully indestructible) is non-iterable. Thus, we can ask:

Is it possible for the first two strcs to be non-sc and still exhibit some form of indestructibility?

The answer is yes.

Theorem 6 *(A., 8/98):*

Let $V \models$ "ZFC + GCH + $\kappa_1 < \kappa_2 < \lambda$ are s.t. κ_1 and κ_2 are both λ sc and λ is strongly inaccessible". There is then a p.o. $\mathbb{P} \in V$, $|\mathbb{P}| = \kappa_2$ s.t. $V_\lambda^{\mathbb{P}} \models$ "κ_1 and κ_2 are both the first two strcs and first two measurables, κ_1's strcness is fully indestructible, and κ_2's measurability (but not necessarily its strcness) is fully indestructible".

Sketch of proof: Assume κ_1, κ_2, and λ are all the least with their properties. By first forcing over V with the p.o. from Theorem 3, we may assume that we have constructed a model $V_1 \models$ "ZFC + κ_1 is the least measurable and is $< \lambda$ strc + κ_1's $< \lambda$ strcness is indestructible under forcing with κ_1-directed closed p.o.s of rank $< \lambda$ + κ_2 is λ sc + λ is the least inaccessible above κ_2".

Working in V_1, we define an Easton support iteration $\mathbb{P}^* = \langle\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha\rangle : \alpha < \kappa_2\rangle$. Let $\langle \delta_\alpha : \alpha < \kappa_2\rangle$ enumerate the measurables in the interval (κ_1, κ_2). Take $\mathbb{P}_0 = \{\emptyset\}$. $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, where if $\Vdash_{\mathbb{P}_\alpha}$ "There is a δ_α-directed closed p.o. of rank below the least inaccessible above δ_α s.t. forcing with it destroys δ_α's measurability", then $\dot{\mathbb{Q}}_\alpha$ is a term for such a p.o. If, however, this is not the case, i.e., if $\Vdash_{\mathbb{P}_\alpha}$ "The measurability of δ_α is indestructible under forcing with δ_α-directed closed p.o.s of rank below the least inaccessible above δ_α", then $\dot{\mathbb{Q}}_\alpha$ is a term for the p.o. adding a non-reflecting stationary set of ordinals of cofinality κ_1 to δ_α.

Why does $\mathbb{P}^*$ work? Its definition ensures it destroys all V_1-measurables in (κ_1, κ_2) and creates no new ones. Since $\mathbb{P}^*$ is κ_1-directed closed, $V_1^{\mathbb{P}^*} \models$ "κ_1 is the least measurable, is $< \lambda$ strc, and the $< \lambda$ strcness of κ_1 is indestructible under forcing with κ_1-directed closed p.o.s of rank $< \lambda$". Suppose $j : V_1 \to M$ is an elementary embedding witnessing the λ-scness of κ_2. If $\Vdash_{\mathbb{P}^*}$ "$\dot{\mathbb{Q}}$ is κ_2-directed closed and $|\dot{\mathbb{Q}}| < \lambda$" and $V^{\mathbb{P}^* * \dot{\mathbb{Q}}} \models$ "κ_2 isn't measurable", then since λ is the least inaccessible above κ_2 in both V and M, at stage κ_2 in the definition of $j(\mathbb{P}^*)$, $\dot{\mathbb{Q}}_{\kappa_2}$ is a term for a p.o. of rank $< \lambda$ destroying κ_2's measurability. By the λ-closure of M wrto V_1, in V_1, $\Vdash_{\mathbb{P}^*}$ "$\dot{\mathbb{Q}}_{\kappa_2}$ destroys κ_2's measurability". We can then use the usual reverse Easton argument to show $j : V_1 \to M$ extends after forcing with $\mathbb{P}^* * \dot{\mathbb{Q}}_{\kappa_2}$, i.e., $V^{\mathbb{P}^* * \dot{\mathbb{Q}}_{\kappa_2}} \models$ "κ_2 is measurable". This contradiction allows us to infer $V_1^{\mathbb{P}^*} \models$ "The measurability of κ_2 is indestructible when forcing with p.o.s of rank $< \lambda$ which are κ_2-directed closed". We thus know that $\dot{\mathbb{Q}}_{\kappa_2}$ is a term for the p.o. adding a non-reflecting stationary set of ordinals of cofinality κ_1 to κ_2. This allows us to use an argument of Magidor to show $V_1^{\mathbb{P}^*} \models$ "κ_2 is λ strc".

There are still many questions remaining concerning strcness and indestructibility. Some are:

1. Can κ_2 above have its strcness fully indestructible?

2. In general, can we have the first α strcs non-sc and fully indestructible?

3. If we don't wish to assume scness, how much indestructibility can be forced for an arbitrary, strc card.?

Note that work of Hamkins shows that if an arbitrary non-sc strc card. is to be made fully indestructible, the p.o. used can't in any way resemble Laver's p.o., i.e., it can't be an Easton support iteration of p.o.s which have the appropriate amount of strategic closure.