Some periodic and non-periodic recursions, Marianna Csornyei (UCL)

It is well known that the recurrence relations

$$x_n = \frac{1}{x_{n-1}}, \quad x_n = \frac{1+x_{n-1}}{x_{n-2}} \quad \text{and} \quad x_n = \frac{1+x_{n-1}+x_{n-2}}{x_{n-3}}$$

are periodic, in the sense that they define periodic sequences for all choices of the initial data, and lead to sequences with periods 2, 5 and 8, respectively. We determine all periodic recursions of the form

$$x_n = \frac{a_0 + a_1 x_{n-1} + \ldots + a_k x_{n-k}}{x_{n-k-1}}, \tag{R}$$

where $a_0, \ldots, a_k$ are complex numbers, $a_1, \ldots, a_{k-1}$ are non-zero and $a_k = 1$.

The non-periodicity of (R) depends on the connection between (R) and the recurrence relations

$$x_n = \max(x_{n-1}, \ldots, x_{n-k}) - x_{n-k-1} \tag{A}$$

and

$$x_n = \max(x_{n-1}, \ldots, x_{n-k}, 0) - x_{n-k-1}. \tag{B}$$

We investigate these recursions together with the related

$$x_n = |\max(x_{n-1}, \ldots, x_{n-k})| - x_{n-k-1}. \tag{C}$$

Each of (A), (B), and (C) leads to periodic sequences if $k = 1$. Also, for $k = 2$, (B) leads to a periodic sequence. However, we prove that no other cases give rise to periodicity.