On uniform Eberlein compacta and c-algebras

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Abstract

We investigate a question of Y. Benyamini, M. E. Rudin and M. Wage, on the existence of universal uniform Eberlein compacta of a given weight, more exactly the related question of the existence of a universal c-algebra of a given size. We show that for any regular $\lambda > \aleph_1$ with $2^{\aleph_0} > \lambda$, there is no c-algebra of size λ universal under c-embeddings. In fact, under these circumstances, for no $\mu < 2^{\aleph_0}$ are there μ c-algebras of size λ such that every c-algebra of size λ c-embeds into one of the μ given ones. [1]

1 Introduction.

We investigate the question of the existence of universal uniform Eberlein compacta. The question of the existence of universal uniform Eberlein compacta of a given size was asked by Y. Benyamini, M.E. Rudin and M. Wage as Problem 3 in their 1977 paper [BeRuWa]. In his recent paper [Be], M. Bell showed that this is equivalent to the existence of universal c-algebras under ordinary embeddings. He showed in [Be] that if $\lambda = 2^{<\lambda}$, then there is

[1]The author thanks Saharon Shelah for bringing M. Bells' [Be] to her attention. This paper is dedicated to the memory of Amer Bešlagić.

a c-algebra of size λ which is universal not just under ordinary embeddings, but also under a stronger notion of a c-embedding. We are interested in the question of the existence of such an algebra when the relevant instances of GCH fail. We show that for any regular cardinal $\lambda > \aleph_1$ with $2^{\aleph_0} > \lambda$, there can be no c-algebra of size λ into which every c-algebra of size λ c-embeds. In fact, under these circumstances, for no $\mu < 2^{\aleph_0}$ are there μ c-algebras of size λ such that every c-algebra of size λ embeds into one of the μ given ones. The niceness property of the c-algebras is not used in the above. We also show that $\clubsuit + \neg CH$ implies that there is no c-algebra of size $\aleph_1$ into which every c-algebra of size $\aleph_1$ c-embeds, in fact, there are no $\mu < 2^{\aleph_0}$ c-algebras of size $\aleph_1$ such that every c-algebra of size $\aleph_1$ c-embeds into one of them. In fact, $\clubsuit_{\rm club}$ suffices in the place of $\clubsuit$. It is a result of M. Bell from [Be] that adding $\aleph_2$ Cohen reals suffices to have a model in which there is no universal uniform Eberlein compact of weight $\aleph_1$, hence also no c-algebra of size $\aleph_1$ universal even under ordinary embeddings.

A uniform Eberlein compact, abbreviated UEC, is a topological space homeomorphic to a weakly compact subspace of a Hilbert space. A UEC X^* is said to be *universal* of weight λ iff every UEC of weight $\leq \lambda$ is a continuous image of it. The intuition suggesting that a universal object is the one into which all other objects of that kind embed is justified when one passes to the objects roughly dual to the UEC, so called c-algebras. Calling a c-algebra B^* universal iff every other c-algebra of the same size embeds into B^*, it turns out that there is a universal UEC of weight λ iff there is a universal c-algebra of size λ (M. Bell in [Be]). Note that this result does not follow immediately from the Stone duality theorem, as not every UEC is 0-dimensional.

For a history and further references on UEC, see papers [BeRuWa] and [Be]. The method of invariants used in this paper was developed in the context of linear orders by M. Kojman and S. Shelah in [KjSh 409], and they have used it in other contexts elsewhere.

2 Preliminaries.

Notation 2.1. For $\alpha > \theta = \mathrm{cf}(\theta)$, we let

$$S^\alpha_\theta \stackrel{\mathrm{def}}{=} \{\beta < \alpha : \mathrm{cf}(\beta) = \theta\}.$$

Definition 2.2. (1) A boolean algebra B is a c-*algebra* iff there is a family $\{B_n : n < \omega\}$ of subsets of B such that

(i) $n \neq m \Longrightarrow B_n \cap B_m = \emptyset$,

(ii) Each B_n consists of pairwise disjoint elements,

(iii) $\bigcup_{n<\omega} B_n$ generates B,

(iv) $\bigcup_{n<\omega} B_n$ has *the nice property*, meaning that for no finite $F \subseteq \bigcup_{n<\omega} B_n$ do we have $\bigvee F = 1$.

We say that $\langle B_n : n < \omega \rangle$ witnesses that B is a c-algebra. When discussing c-algebras, we always have in mind a fixed sequence witnessing this, although we may omit to mention it. We may refer to it as $\langle B_n(B) : n < \omega \rangle$.

(2) If B^*_l for $l \in \{0, 1\}$ are c-algebras, then a 1-1 boolean homomorphism $f : B^*_0 \to B^*_1$ is a c-*embedding* iff $f``B_n(B^*_0) \subseteq B_n(B^*_1)$ for all $n < \omega$. If there is such an embedding from B^*_0 to B^*_1, we write

$$B^*_0 < B^*_1.$$

(3) A c-algebra B^* of size λ is *universal* λ iff for any c-algebra B of size $\leq \lambda$, there is an embedding $f : B \to B^*$.
(4) A c-algebra B^* of size λ is *c-universal* λ iff for any c-algebra B of size $\leq \lambda$, there is a c-embedding $f : B \to B^*$.
(5) A boolean algebra is *almost c* iff it satisfies all the properties of c-algebras, except possibly for (iv) in (1) above.

Fact 2.3. (M. Bell, [Be]) (1) There is a universal UEC of size λ iff there is a universal c-algebra of size λ.
(2) If $\lambda = 2^{<\lambda}$, then there is a c-universal c-algebra of size λ.

Definition 2.4. Let $\lambda = \mathrm{cf}(\lambda) > \aleph_0$.
(1) A sequence $\langle c_\delta : \delta \in S \rangle$ is called a *club-sequence* iff S is a stationary set of limit ordinals $< \lambda$, and for each $\delta \in S$, the set c_δ is a club subset of δ.
(2) A club sequence $\langle c_\delta : \delta \in S \rangle$ is called *a club-guessing sequence* iff for every club E of λ there is a stationary set of $\delta \in S$ such that $c_\delta \subseteq E$.

The following fact is a result of S. Shelah from [Sh e] [III 7.8], see also [Sh g], and a proof is also in M. Kojman and S. Shelah's [KjSh 409].

Fact 2.5. (S. Shelah, [Sh e]) Suppose $\lambda = \mathrm{cf}(\lambda) > \aleph_1$.
<u>Then</u> there is a club guessing sequence $\langle c_\delta : \delta \in S \subseteq S^\lambda_{\aleph_0} \rangle$.

Definition 2.6. (1) $\clubsuit$ is the statement that there is a sequence

$$\langle c_\delta : \delta \text{ limit } < \omega_1 \rangle$$

with $c_\delta \subseteq \delta = \sup(c_\delta)$ such that for every $A \subseteq \omega_1$ unbounded,

$$\{\delta : c_\delta \subseteq A\}$$

is stationary.
(2) $\clubsuit_{\mathrm{club}}$ is the statement obtained from $\clubsuit$ by replacing above in "every $A \subseteq \omega_1$ unbounded", the word "unbounded" by "club".

3 Non-existence of c-universal c-algebras.

Theorem 3.1. Suppose that $\lambda = \mathrm{cf}(\lambda) > \aleph_1$ satisfies $2^{\aleph_0} > \lambda$.
<u>Then</u> there is no c-universal c-algebra of size λ, moreover for no $\mu < 2^{\aleph_0}$ are there μ c-algebras of size λ such that every c-algebra of size $\leq \lambda$ c-embeds into one of them.

Proof. Fix λ as in the hypothesis, and a club guessing sequence

$$\bar{c} = \langle c_\delta : \delta \in S \subseteq S^\lambda_{\aleph_0} \rangle$$

as guaranteed by Fact 2.5.

Definition 3.2. (1) Suppose that B is a boolean algebra of size λ. A sequence $\bar{B} = \langle B^\alpha : \alpha < \lambda \rangle$ is a *filtration of* B iff

(i) $B^\alpha \subseteq B^{\alpha+1}$ and $B^0 = \emptyset$.

(ii) $B^\delta = \bigcup_{\alpha<\delta} B^\alpha$ for δ limit,

(iii) $|B^\alpha| < \lambda$,

(iv) $\bigcup_{\alpha<\lambda} B^\alpha = B$.

(2) Suppose that B is a c-algebra of size λ with a filtration $\bar{B}$, while $\delta \in S$ and $b \in B \setminus B^\delta$. We define

$$\mathrm{Inv}_{\bar{B}}(b, c_\delta) \stackrel{\text{def}}{=} \left\{\alpha \in c_\delta : (\exists m \geq 1)(\exists y \in B_m(B) \cap B^{\min(c_\delta \setminus (\alpha+1))} \setminus B^\alpha)\, [y \geq b]\right\}.$$

If $\alpha \in \mathrm{Inv}_{\bar{B}}(b, c_\delta)$ because $m \geq 1$ is such that for some $y \geq b$ we have $y \in B_m(B) \cap B^{\min(c_\delta \setminus (\alpha+1))} \setminus B^\alpha$, we say that $\alpha \in \mathrm{Inv}_{\bar{B}}(b, c_\delta)$ by virtue of m.

Note 3.3. With the notation of Definition 3.2:
(1) For every $m \geq 1$, there is at most one $\alpha \in \mathrm{Inv}_{\bar{B}}(b, c_\delta)$ which is there by virtue of m (unless $b = 0$).

[Why? As elements of $B_m(B)$ are pairwise disjoint.]
(2) $|\{A : (\exists b \in B)(\exists \delta \in S)\mathrm{Inv}_{\bar{B}}(b, c_\delta) = A\}| \leq \lambda$.

Main Claim 3.4. Suppose that for $\delta \in S$ we are given $A_\delta \subseteq c_\delta$ with $\sup(A_\delta) = \delta$ and $\mathrm{otp}(A_\delta) = \omega$.

<u>Then</u> there is a c-algebra B and filtration $\bar{B}$ of B such that for all $\delta \in S$ there is $b^0_\delta \in B \setminus B^\delta$ such that $\mathrm{Inv}_{\bar{B}}(b^0_\delta, c_\delta) = A_\delta$.

Proof of the Main Claim. Let $f_\delta : \omega \to A_\delta$ be the increasing enumeration of A_δ. We shall use α^δ_m to stand for $f_\delta(m)$.

For $n < \omega$, let X_n be the boolean algebra generated by $\{a_\eta : \eta \in {}^{\leq n}\lambda\}$ freely except for the equations

(1) $a_{\langle\rangle} = 1$,

(2) $[\eta \neq \varepsilon \ \&\ \lg(\eta) = \lg(\varepsilon)] \Longrightarrow a_\eta \wedge a_\varepsilon = 0$,

(2) $\eta \trianglelefteq \varepsilon \Longrightarrow a_\eta \geq a_\varepsilon$.

Let $\mathcal{B}^*$ be the product algebra $\langle X_n : n < \omega \rangle$. Let I be the ideal of eventually 0 elements of $\mathcal{B}^*$, and let $\mathcal{B} \stackrel{\text{def}}{=} \mathcal{B}^*/I$.

For $\delta \in S$ let:

$$b^0_\delta \stackrel{\text{def}}{=} [\langle a_{f_\delta \restriction 0}, a_{f_\delta \restriction 1}, \ldots a_{f_\delta \restriction n}, \ldots \rangle]$$

and for $m \geq 1$ let b^m_δ be the class of the function in $\mathcal{B}^*$ which is constantly equal to $a_{f_\delta \restriction m}$. Note that $b^m_\delta \geq b^0_\delta$ for all $m \geq 1$. Also note that b^0_δ are pairwise disjoint, as for $\delta_1 \neq \delta_2$ both in S, we have that $A_{\delta_1} \cap A_{\delta_2}$ is finite.

For $n < \omega$ let $B_n \stackrel{\text{def}}{=} \{b^n_\delta : \delta \in S\}$. Our algebra B is given by

$$B \stackrel{\text{def}}{=} \langle \bigcup_{n<\omega} B_n \rangle_{\mathcal{B}}.$$

It is easily seen that B is a c-algebra of size λ.

Next we define the filtration $\bar{B} = \langle B^\alpha : \alpha < \lambda \rangle$ of B by letting $B^0 = \emptyset$ and for $\alpha < \lambda$

$$B^{\alpha+1} \stackrel{\text{def}}{=} \left\langle \bigcup_{n<\omega} B_n \cap \{[\bar{b}] : (\forall n)(\exists \eta \in {}^{\leq n}\alpha)\, [(b(n) = a_\eta \text{ or } b(n) = 0]\} \right\rangle_B,$$

while $B^\alpha \stackrel{\text{def}}{=} \bigcup_{\beta<\alpha} B^\beta$ for α limit.

Clearly, $\bar{B}$ is a filtration of B, and $b^0_\delta \in B \setminus B^\delta$. The proof will be finished once we prove:

Subclaim 3.5. For $\delta \in S$ we have $\mathrm{Inv}_{\bar{B}}(b^0_\delta, c_\delta) = A_\delta$.

Proof of the Subclaim. Note that for $\delta \in S$ and $m \geq 1$ we have $b^m_\delta \in B^{\min(c_\delta \setminus (\alpha^\delta_{m-1}+1))} \setminus B^{\alpha^\delta_{m-1}}$, hence $\alpha^\delta_{m-1} \in \mathrm{Inv}_{\bar{B}}(b^0_\delta)$ by virtue of m. Now use Note 3.3(1). $\bigstar_{3.5}$ $\bigstar_{3.4}$

Claim 3.6. Suppose that $\bar{A} = \langle A_\delta : \delta \in S \rangle$ is as in the statement of the Main Claim, and $B = B[\bar{A}]$ and $\bar{B}$ are obtained as in the Main Claim. Further suppose that $f : B \to B^*$ is a c-embedding,while $\bar{B}^* = \langle B^\alpha_* : \alpha < \lambda \rangle$ is a filtration of B^* (so $|B^*| = \lambda$).

Then there is a club E of λ such that for every $\delta \in S$ with the property $c_\delta \subseteq E$ we have

$$A_\delta = \mathrm{Inv}_{\bar{B}}(b^0_\delta, c_\delta) = \mathrm{Inv}_{\bar{B}^*}(f(b^0_\delta), c_\delta).$$

Proof of the Claim. Let

$$E \stackrel{\mathrm{def}}{=} \{\alpha < \lambda : B^\alpha_* \cap f``B = f``B^\alpha\}.$$

Hence E is a club of λ.

Suppose now that $\delta \in S$ and $c_\delta \subseteq E$. Clearly

$$f(b^m_\delta) \in B_*^{\min(c_\delta \setminus (\alpha^\delta_{m-1}+1))} \setminus B_*^{\alpha^\delta_{m-1}},$$

by the definition of E, and the rest follows as in the proof of $A_\delta = \mathrm{Inv}_{\bar{B}}(b^0_\delta, c_\delta)$. $\bigstar_{3.6}$

Proof of the Theorem finished. Suppose that B^* is a c-algebra of size λ and $\bar{B}^*$ any filtration of B^*. As $\lambda < 2^{\aleph_0}$, we can choose for each $\delta \in S$ an unbounded $A_\delta \subseteq c_\delta$ of order type ω, such that for no $b \in B^* \setminus B^\delta_*$ do we have $\mathrm{Inv}_{\bar{B}^*}(b, c_\delta) = A_\delta$. Let $\bar{A} \stackrel{\mathrm{def}}{=} \langle A_\delta : \delta \in S\rangle$ and $B \stackrel{\mathrm{def}}{=} B[\bar{A}]$.

Suppose $f : B \to B^*$ is a c-embedding. Let E be a club of λ guaranteed by Claim 3.6, and let $\delta \in S$ be such that $c_\delta \subseteq E$. Hence $\mathrm{Inv}_{\bar{B}^*}(f(b^0_\delta), c_\delta) = A_\delta$, a contradiction.

The part of the theorem involving $\mu < 2^{\aleph_0}$ is proved similarly. $\bigstar_{3.1}$

Theorem 3.7. (1) Suppose that $\clubsuit_{\mathrm{club}} + \neg CH$ holds.

Then there is no c-universal c-algebra of size $\aleph_1$. Moreover for no $\mu < 2^{\aleph_0}$ are there μ c-algebras of size $\aleph_1$ such that every c-algebra of size $\leq \aleph_1$ c-embeds into one of them.

(2) Suppose that either $\lambda = \mathrm{cf}(\lambda) > \aleph_1$ satisfies $2^{\aleph_0} > \lambda$, or $\lambda = \aleph_1$ and $\clubsuit_{\mathrm{club}} + \neg CH$ holds.

Then there is no almost c-algebra of size λ which is c-universal for almost c-algebras of size $\leq \lambda$, or even for c-algebras of size $\leq \lambda$. Moreover no $\mu < 2^{\aleph_0}$ almost c-algebras of size λ taken together have that property. $\bigstar_{3.7}$

Proof. The same proof as that of Theorem 3.1.

References

[Be] M. Bell, *Universal Uniform Eberlein Compact Spaces*, submitted.

[BeRuWa] Y. Benyamini, M.E. Rudin and M. Wage, *Continuous Images of Weakly Compact Subsets of Banach Spaces*, Pacific Journal of Mathematics, vol. 70, No. 2, 1977, pg. 309-324.

[KjSh 409] M. Kojman and S. Shelah, *Nonexistence of universal orders in many cardinals*, Journal of Symbolic Logic 57 (1992), 875-891.

[KjSh 455] M. Kojman and S. Shelah, *Universal Abelian Groups*, Israel Journal of Mathematics, 1995.

[Sh g] S. Shelah, *Cardinal Arithmetic*, Oxford University Press 1994.

[Sh e] S. Shelah, *Universal classes*, in preparation.

[Sh 175] S. Shelah, *Universal graphs without instances of CH*, Annals of Pure and Applied Logic vol. 26 (1984), pg. 75-87.

[Sh 175A] S. Shelah, *Universal graphs without instances of CH: Revisited*, Israel Journal of Math, vol. 70, No.1 (1990), pg. 69-81.