TSIRELSON SPACES AND TURBULENT ACTIONS

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Let X be a complete separable metric space. If E, F are Borel equivalence relations on X (i.e. the graphs of E, F are Borel subsets of X^2), then

$$E \leq F$$

means that there is a Borel-measurable map $g\colon X \to X$ such that for all x, y[2]

$$x \, E \, y \iff g(x) \, F \, g(y).$$

Then $E \leq F$ can be interpreted as:
1) *The Borel-cardinality of X/E is smaller than the Borel-cardinality of X/F*, or as
2) *E is simpler than F*.

[2] $x \, E \, y$ stands for $\langle x, y \rangle \in E$

Two dichotomies

The equality relation on X is denoted by X. E shall always denote a Borel equivalence relation.

Silver (1980): $E \leq \mathbb{N}$ or $\mathbb{R} \leq E$.

Let E_0 be the relation on $\mathcal{P}(\mathbb{N})$, defined by

$$X \, E_0 \, Y \iff X \Delta Y \text{ is finite.}$$

Note that E_0 is an F_σ relation if $\mathcal{P}(\mathbb{N})$ is considered with its Cantor-set topology.

Harrington–Kechris–Louveau, 1990:
$E \leq \mathbb{R}$ or $E_0 \leq E$.

Therefore the initial segment of $\leq$ on the Borel equivalence relations looks as follows:

$$1 \leq 2 \leq \ldots \leq \mathbb{N} \leq \mathbb{R} \leq E_0.$$

An equivalence relation E is "simply classifiable" if we can assign simple invariants to its equivalence classes in an effective way.

Examples of "simple invariants":
1) Cantor–Bendixson rank+size of the top level (E is the homeomorphism relation on closed countable sets of reals),
2) Ulm invariants (E is the isomorphism on countable torsion Abelian groups)

Hjorth, 1996: If an orbit equivalence relation is *turbulent*, then it can not be effectively classified by using countable structures.

Def. *(Hjorth) An action of a group G on a space X is turbulent if*
1) every orbit is dense,
2) every orbit is meager, and
3) for every neighbourhood V of e_G, for every open $U \subseteq X$, for every $x \in U$ and $y \in X$, there are $g_n \in V$ such that the sequence

$$x, \quad g_1 x, \quad g_2 g_1 x, \quad g_3 g_2 g_1 x, \quad g_4 g_3 g_2 g_1 x, \ldots$$

is included in U and has a subsequence converging to some $y_0 \in [y]_G$.

The induced equivalence relation is turbulent if the action is.

($[y]_G$ is the orbit of y.)

Relations induced by ideals

If $\mathcal{I}$ is a (Borel) ideal on $\mathbb{N}$, then $E_{\mathcal{I}}$:

$$X \, E_{\mathcal{I}} \, Y \iff X \Delta Y \in \mathcal{I}$$

is a (Borel) equivalence relation on $\mathcal{P}(\mathbb{N})$. Note
1) $E_0 = E_{\mathrm{Fin}}$.
2) $E_{\mathcal{I}}$ is the coset equivalence relation of $\mathcal{I}$ on $\langle \mathcal{P}(\mathbb{N}), \Delta \rangle$ ($\approx \prod_{i=1}^{\infty} \mathbb{Z}_2$).

A preordering on the ideals:
$\mathcal{I} \leq_{RB}^{++} \mathcal{J}$ iff there are $u_1 < u_2 < u_3 < \cdots \subseteq \mathbb{N}$ such that

$$A \in \mathcal{I} \qquad \text{iff} \qquad \bigcup_{i \in A} u_i \in \mathcal{J}.$$

Fact. $\mathcal{I} \leq_{RB}^{++} \mathcal{J}$ implies $E_{\mathcal{I}} \leq E_{\mathcal{J}}$.

Pf. The map $A \mapsto \bigcup_{i \in A} u_i$ is a continuous reduction. $\square$

Mathias: $\mathrm{Fin} \leq_{RB}^{++} \mathcal{I}$ for every analytic ideal $\mathcal{I}$.

Lemma $E_{\mathcal{I}}$ is turbulent if $\mathcal{I}$ is *dense*, i.e. if $\forall A \in \mathcal{P}_\infty(\mathbb{N})\, \exists B \in \mathcal{P}_\infty(A)(B \in \mathcal{I})$.

Examples of Borel ideals:

$$\mathrm{Fin} = \text{all finite subsets of } \mathbb{N}$$

$$\mathcal{Z}_0 = \{A : \limsup_n \frac{|A \cap \{1, \ldots, n\}|}{n} = 0\}$$

$$\mathcal{I}_{1/n} = \{A : \sum_{n \in A} \frac{1}{n} < \infty\}$$

$$\mathcal{I}_{\mathrm{GP}} = \text{ the ideal generated by geometric progressions}\}$$

$$\mathcal{I}_{\mathrm{GP}}^{\perp} = \{A : A \cap P \text{ is finite for every geometric progression } P\}$$

These ideals under the $\leq_{RB}^{++}$ ordering:

$$
\begin{tikzcd}
 & \mathcal{Z}_0 & \\
\mathcal{I}_{GP} & \mathcal{I}_{GP}^{\perp} \arrow[u] & \mathcal{I}_{1/n} \\
 & \mathrm{Fin} \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

… and their equivalence relations under the $\leq$ ordering:

$$
\begin{tikzcd}
 & E_{\mathcal{Z}_0} & \\
E_{\mathcal{I}_{GP}} & E_{\mathcal{I}_{GP}^{\perp}} \arrow[u] & E_{\mathcal{I}_{1/n}} \\
 & E_0 = E_{\mathrm{Fin}} \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

Def. *An ideal $\mathcal{I}$ is a P-ideal if it is σ-directed under the inclusion modulo finite.*

The ideals Fin, $\mathcal{Z}_0, \mathcal{I}_{1/n}$ and $\mathcal{I}_{GP}^{\perp}$ are P-ideals, but $\mathcal{I}_{GP}$ is not.

Solecki: If $\mathcal{I}$ is analytic, then $\mathcal{I}$ is a P-ideal iff the group $\langle \mathcal{I}, \Delta \rangle$ is Polishable.

Solecki: For every analytic $\mathcal{I}$, if $E_{\mathcal{I}} \not\leq E_0$, then
1) $\mathcal{I}_{GP} \leq_{RB}^{++} \mathcal{I}$ iff $\mathcal{I}$ is not a P-ideal,
2) If $\mathcal{I}$ is a P-ideal and not F_σ, then $\mathcal{I}_{GP}^{\perp} \leq_{RB}^{++} \mathcal{I}$.

$$\begin{tikzcd}
E_{(\text{non-P})} & E_{(\text{P+non-}F_\sigma)} & \\
E_{\mathcal{I}_{GP}} \arrow[u] & E_{\mathcal{I}_{GP}^{\perp}} \arrow[u] & E_{(F_\sigma \text{ P-ideal})} \\
& E_0 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

F_σ P-ideals

$$\mathcal{I}_{1/n} = \{A : \sum_{n \in A} \frac{1}{n} < \infty\}, \quad \text{(an } F_\sigma\text{-definition)}$$

$$= \{A : \lim_n \sum_{i \in A \setminus n} \frac{1}{i} = 0\} \quad \text{(a P-definition).}$$

A variation:

$$\mathcal{I}_f = \{A : \sum_{n \in A} f(n) < \infty\}.$$

Fact. $\mathcal{I}_f \approx \mathsf{Fin}$ or $\mathcal{I}_{1/n} \leq^{++}_{RB} \mathcal{I}_f \leq^{++}_{RB} \mathcal{I}_{1/n}$.

Are there any other F_σ P-ideals?

Conjecture (Kechris, Mazur): *For every F_σ P-ideal $\mathcal{I}$ either $E_{\mathcal{I}} \leq E_0$ or $E_{\mathcal{I}_{1/n}} \leq E_{\mathcal{I}}$. Namely (by Solecki's result):*

$$\begin{tikzcd}
E_{(\text{non-P})} & E_{(\text{P+non-}F_\sigma)} & E_{(F_\sigma\ \text{P})} \\
E_{\mathcal{I}_{GP}} \arrow[u] & E_{\mathcal{I}^{\perp}_{GP}} \arrow[u] & E_{\mathcal{I}_{1/n}} \arrow[u] \\
 & E_0 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Ideals and submeasures

Def. *A map $\varphi\colon \mathcal{P}(\mathbb{N}) \to [0,\infty]$ is a submeasure if*
$\varphi(\emptyset) = 0$ *and it is*
monotonic: $\varphi(s) \leq \varphi(t)$, *for all* $s \subseteq t$
subadditive: $\varphi(s \cup t) \leq \varphi(s) + \varphi(t)$, *for all* s, t.

φ is *lower semicontinuous*, or *lsc*, if moreover

$$\lim_n \varphi(A \cap n) = \varphi(A) \qquad \text{for all } A.$$

If φ is a lsc submeasure, then

$$\mathrm{Fin}(\varphi) = \{A : \varphi(A) < \infty\}$$

is an F_σ-ideal, while

$$\mathrm{Exh}(\varphi) = \{A : \lim_n \varphi(A \setminus n) = 0\}$$

is a Borel P-ideal.

Remarkably, the converse is true:

Mazur, Solecki:

$$\mathcal{I} \text{ is} \left\{ \begin{array}{c} \text{an } F_\sigma \\ \text{a Borel P-} \\ \text{an } F_\sigma \text{ P-} \end{array} \right\} \text{ideal iff } \mathcal{I} = \left\{ \begin{array}{c} \mathrm{Fin}(\varphi) \\ \mathrm{Exh}(\varphi) \\ \mathrm{Fin}(\varphi) = \mathrm{Exh}(\varphi) \end{array} \right.$$

for some lsc submeasure φ.

folklore: For every analytic P-ideal $\mathcal{I}$, $E_{1/n} \le E_{\mathcal{I}}$ iff $\mathcal{I}_{1/n} \le^{++}_{RB} \mathcal{I}$.

With a bit of imagination, one can relate this result to:

Pelczynski: A Banach space X includes a copy of ℓ_1 if and only if its basis has a block subsequence equivalent to the standard basis of ℓ_1.

Note that the ideal

$$\mathcal{I}^p = \left\{ A : \left(\sum_{n \in A} \left(\frac{1}{n} \right)^p \right)^{1/p} < \infty \right\} \qquad (p \ge 1)$$

coincides with

$$\mathcal{I}_{(1/n)^p} = \left\{ A : \sum_{n \in A} \left(\frac{1}{n} \right)^p < \infty \right\}.$$

But there is a more informative way to associate analytic P-ideals to a separable Banach space X with an unconditional basis e_n.

Fix a Banach space X with an unconditional basis e_n, and a sequence φ_n of submeasures such that φ_n concentrates on $[2^n, 2^{n+1})$. let

$$x_m = \sum_{n=1}^{m} \varphi_n(A \cap [2^n, 2^{n+1}))e_n$$
$$\varphi(A) = \lim_m \|x_m\|$$
$$\mathcal{J}_X = \mathrm{Exh}(\varphi) = \{A : \lim_n \varphi(A \setminus n) = 0\}$$

With the appropriate φ_n's:
ℓ_1 gives $\mathcal{I}_{1/n}$,
c_0 gives $\mathcal{Z}_0$ (and $\mathcal{Z}_0$ is not F_σ),
ℓ_p gives an F_σ P-ideal $\gneq^{++}_{RB} \mathcal{I}_{1/n}$.

An approach to the Kechris–Mazur conjecture:

1) Fix $\mathcal{I} = \mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$,

2) Assume φ is composed out of φ_n $(n \in \mathbb{N})$,

3) Use Ramsey's theorem to find a uniformized subsequence φ_{n_i},

4) Find $t_i \subseteq \operatorname{supp}(\varphi_{n_i})$ such that

$$\bigcup_{n \in A} t_n \in \mathcal{I} \qquad \text{iff} \qquad A \in \mathcal{I}_{1/n}.$$

Q: *Banach (1930): Does every ∞-dimensional Banach space contain a copy of c_0 or some ℓ_p ($p \geq 1$)?*

If Banach's question had a positive answer, it would not immediately imply a positive answer to KMC; it would, however, give it a strong support. But

Tsirelson (1974): There is an ∞-dimensional Banach space which does not include a copy of c_0 or ℓ_p ($p \geq 1$).

This does not immediately imply that KMC fails, but it does give a new way of constructing submeasures, leading to

Thm 1. There is an F_σ P-ideal $\mathcal{T}$ such that $E_{\mathcal{T}} \not\leq E_0$ and $E_{1/n} \not\leq E_{\mathcal{T}}$.[3]

$$
\begin{tikzcd}
E_{\mathcal{I}_{GP}} & E_{\mathcal{I}_{GP}^{\perp}} & E_{\mathcal{I}_{1/n}} & E_{\mathcal{T}} & ? \\
& E_0 \arrow[ul] \arrow[u] \arrow[ur] & & & \\
& \mathbb{R} \arrow[u] & & & \\
& \mathbb{N} \arrow[u] & & & \\
& 1, 2, 3, \ldots \arrow[u] & & &
\end{tikzcd}
$$

[3]after the completion of this work, we have learned that B. Velickovic has independently proved Thm 1

Tsirelson submeasures

Fix a strictly increasing $h\colon \mathbb{N} \to \mathbb{N}$.

A finite sequence $\vec{A} = \langle A_1, A_2, \ldots, A_k \rangle$ of subsets of $\mathbb{N}$ is *h-admissible* if

a) $A_1 < A_2 < \cdots < A_k$, and

b) $h(\min A_1) \geq k = |\vec{A}|$.

Let $\mathcal{A}_h$ be the set of all h-admissible finite sequences.

$A_1 \quad A_2 \quad \ldots \quad A_{h(n)} \qquad \in \mathcal{A}_h$

n

$B_1 \quad B_2 \quad \ldots \quad B_{h(n)} \qquad \notin \mathcal{A}_h$

n

$C_1 \quad C_2 \quad \ldots \quad C_k \qquad \notin \mathcal{A}_h$

$D_1 \quad D_2 \quad \ldots \quad D_{h(n)-k} \qquad \in \mathcal{A}_h$

n

Fix some "slow enough" $f \in c_0 \setminus \ell_1$.
Define the submeasures $\tau_n = \tau_{fhn}$ by

$$\tau_0(B) = \sup_{i \in B} f(i)$$

$$\tau_{n+1}(B) = \max(\tau_n(B), \frac{1}{2} \sup_{\vec{A} \in \mathcal{A}_h} \sum_{i=1}^{|\vec{A}|} \tau_n(A_i \cap B)),$$

$$\tau(B) = \sup_n \tau_n(B) = \lim_n \tau_n(B)$$

Then the *Tsirelson ideal*

$$\mathcal{T}_{fh} = \mathrm{Fin}(\tau_{fh}) = \{A : \tau_{fh}(A) < \infty\}$$

satisfies the conclusion of Thm 1.

Fact. $\tau(B) \neq \tau_0(B)$ implies

$$\tau(B) = \frac{1}{2} \sup_{\vec{A} \in \mathcal{A}_h} \sum_{i=1}^{|\vec{A}|} \tau(A_i \cap B)).$$

Lemma $\mathcal{T}$ is a P-ideal.

Pf. It suffices to prove $\mathrm{Exh}(\tau) \supseteq \mathrm{Fin}(\tau)$. If $A \notin \mathrm{Exh}(\tau)$, then we can find $\varepsilon > 0$ and

$$s_i \subseteq A, \qquad \tau(s_i) \geq \varepsilon > 0,$$

$$s_1 < s_2 < s_3 < \cdots < k \leq s_n < s_{n+1} < \ldots$$

so that, since $\langle s_n, \ldots, s_{n+h(k)} \rangle$ is h-admissible,

$$\tau(A) \geq \tau\big(s_n \cup s_{n+1} \cup \cdots \cup s_{n+h(k)}\big) \geq \frac{h(k)\varepsilon}{2} \to \infty.$$

Casazza–Johnson–Tzafriri: If $\mathcal{I} \leq^{++}_{RB} \mathcal{T}_{fh}$, then $\mathcal{I}$ is a Tsirelson ideal.

Since $E_{1/n} \leq E_{\mathcal{T}}$ implies $\mathcal{I}_{1/n} \leq^{++}_{RB} \mathcal{T}$, the above implies

Corollary. If $E_{1/n} \leq E_{\mathcal{T}}$, then $\mathcal{I}_{1/n}$ is equal to some Tsirelson ideal.

Therefore the proof of Theorem 1 reduces to

Lemma $\mathcal{I}_{1/n} \neq \mathcal{T}_{fh}$, for all f, h.

Basis problem for turbulent actions

Def. *Turbulent orbit equivalence relations $F_1, \ldots, F_2$ form a basis if for every other turbulent orbit equivalence relation E we have $F_i \leq E$ for some i.*

Q (Hjorth, Kechris) *Is there a finite basis for turbulent orbit equivalence relations?*

Two turbulent equivalence relations F_1, F_2 are *incompatible* if there is no turbulent $E \leq F_1, F_2$.

Fact. If $F_1, \ldots, F_k$ are pairwise incompatible, then every basis for turbulent relations has to be of size at least k.

Hjorth: $E_{\mathcal{Z}_0}$ and $E_{1/n}$ are incompatible turbulent orbit equivalence relations.

Thm 2. $E_{\mathcal{T}}$ is incompatible with both $E_{\mathcal{I}_{1/n}}$ and $E_{\mathcal{Z}_0}$.

Conjecture. *For every* $h_1\colon \mathbb{N} \to \mathbb{N}$ *there is* $h_2 >> h_1$ *such that* $\mathcal{T}_{f h_2} \not\leq_B \mathcal{T}_{f h_1}$.

By a recursive construction, this would give a transfinite strictly decreasing chain of turbulent relations.

But how do we know there is no turbulent orbit equivalence relation below this chain?

$\approx$**Hjorth:** If E is turbulent and reducible to $E_{\mathcal{I}}$ for some F_σ P-ideal $\mathcal{I}$, then there is a turbulent ideal $\mathcal{J} \leq^{++}_{RB} \mathcal{I}$ such that $E_{\mathcal{J}} \leq E$.

$$\begin{tikzcd}
 & E_{\mathcal{I}} \\
E \arrow[ur] & \\
 & E_{\mathcal{J}} \arrow[uu] \arrow[ul]
\end{tikzcd}$$

Since $\mathcal{J} \leq^{++}_{RB} \mathcal{T}_{fh}$ implies $\mathcal{J}$ is Tsirelson, the above Conjecture would imply a negative answer to Hjorth–Kechris question.

Change the price

Fix an $\alpha \in (0,1)$,
a strictly increasing $h\colon \mathbb{N} \to \mathbb{N}$, and
a "slow enough" $f \in c_0 \setminus \ell_1$.

Define $\tau_{\alpha n} = \tau_{fh\alpha n}$ $(n = 0, 1, \dots)$ by

$$\tau_{\alpha 0}(B) = \sup_{i \in B} f(i)$$

$$\tau_{\alpha n+1}(B) = \max(\tau_{\alpha n}(B), \alpha \sup_{\vec{A} \in \mathcal{A}_h} \sum_{i=1}^{|\vec{A}|} \tau_{\alpha n}(A_i \cap B)),$$

$$\tau_{\alpha}(B) = \sup_n \tau_{\alpha n}(B) = \lim_n \tau_{\alpha n}(B)$$

Then an α-Tsirelson ideal

$$\mathcal{T}_{fh\alpha} = \mathrm{Fin}(\tau_\alpha) = \{A : \tau_\alpha(A) < \infty\}$$

is an F_σ P-ideal.

Note that $\mathcal{T}_{fh\alpha} = \mathcal{T}_{fh1/2}$.

Thm 3. If $\alpha \neq \beta$, then $E_{\mathcal{T}_\alpha}$ and $E_{\mathcal{T}_\beta}$ are incompatible.

Corollary. There is a set of continuum many incompatible turbulent orbit equivalence relations.

Corollary. Every basis for turbulent orbit equivalence relations is of size $\mathfrak{c}$.

The proof of Thm 3 is parallel to that of Thm 1:

Lemma If $\mathcal{I} \leq_{RB}^{++} \mathcal{T}_{fh\alpha}$, then $\mathcal{I}$ is also an α-Tsirelson ideal.

Lemma If $\alpha \neq \beta$ and $\mathcal{T}_{fh\alpha}$, $\mathcal{T}_{f'h'\beta}$ are turbulent, then $\mathcal{T}_{fh\alpha} \neq \mathcal{T}_{f'h'\beta}$.

By $\approx$Hjorth's theorem, it suffices to take care on the relations induced by ideals, and this completes the proof.

Q: *Is there a turbulent orbit equivalence relation without a minimal turbulent orbit equivalence relation below it?*

($E_{\mathcal{T}_{fh}}$, for example?)

Q: *Is there a minimal turbulent orbit equivalence relation other than $E_{\mathcal{I}_{1/n}}$?*

Perhaps Schlumprecht's space can be used to define one?

Q: *Are the ideals $\mathcal{T}$ and $\mathcal{I}_{1/n}$ cofinally (i.e. Tukey) equivalent?*