IDEALS INDUCED BY TSIRELSON SUBMEASURES

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Abstract. We use Tsirelson's Banach space ([2]) to define an F_σ P-ideal which refutes a conjecture of Mazur and Kechris (see [12, 9, 8]).

1. Introduction

By the dichotomy results of Silver and Harrington–Kechris–Louveau (see [10, 8]), the Borel-cardinality of quotients over Borel equivalence relations on Polish spaces is well-understood below $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$. This can not be said for the next level of this ordering, even if we restrict our attention to Borel-cardinalities of quotients $\mathcal{P}(\mathbb{N})/\mathcal{I}$ over Borel ideals $\mathcal{I}$. The two natural 'successors' of Fin are the Fubini ideals on $\mathbb{N}^2$: $\mathrm{Fin}\times\emptyset$ (also called I_1) consisting of all sets with only finitely many nonempty vertical sections, and $\emptyset\times\mathrm{Fin}$ (also called I_3 and Fin^ω) consisting of all sets all of whose vertical sections are finite. By results of Solecki ([17]), quotients over these two ideals are the critical points for quotients over Borel ideals which are not P-ideals and for Borel P-ideals which are not F_σ, respectively ($\mathcal{I}$ is a *P-ideal* if it is σ-directed under the inclusion modulo finite). In [9], Kechris posed the following trichotomy conjecture for Borel ideals $\mathcal{I}$ such that $\mathcal{P}(\mathbb{N})/\mathcal{I}\not\leq_B\mathcal{P}(\mathbb{N})/\mathrm{Fin}$: at least one of $\mathcal{P}(\mathbb{N}^2)/\mathrm{Fin}\times\emptyset$, $\mathcal{P}(\mathbb{N}^2)/\emptyset\times\mathrm{Fin}$, and $\mathcal{P}(\mathbb{N})/\mathcal{I}_{1/n}$ is $\leq_B\mathcal{P}(\mathbb{N})/\mathcal{I}$ (the summable ideal $\mathcal{I}_{1/n}$ is defined below). By the above results of Solecki, this is equivalent to an earlier dichotomy conjecture of Mazur ([12]): If $\mathcal{I}$ is an F_σ ideal such that $\mathcal{P}(\mathbb{N})/\mathcal{I}\not\leq_B\mathcal{P}(\mathbb{N})/\mathrm{Fin}$, then either $\mathcal{P}(\mathbb{N}^2)/\mathrm{Fin}\times\emptyset$ or $\mathcal{P}(\mathbb{N})/\mathcal{I}_{1/n}$ is $\leq_B\mathcal{P}(\mathbb{N})/\mathcal{I}$.

Consider an ordering on Borel ideals simpler than $\leq_B$:

$$\mathcal{I}\leq^+_{RB}\mathcal{J} \text{ if there is } A\subseteq\mathbb{N} \text{ and } h\colon A\to\mathbb{N} \text{ such that } B\in\mathcal{I} \text{ iff } h^{-1}(B)\in\mathcal{J}.$$

If $A=\mathbb{N}$, then we write $\mathcal{I}\leq_{RB}\mathcal{J}$. Clearly $\mathcal{I}\leq^+_{RB}\mathcal{J}$ implies $\mathcal{P}(\mathbb{N})/\mathcal{I}\leq_B\mathcal{P}(\mathbb{N})/\mathcal{J}$, as the mapping $A\mapsto h^{-1}(A)$ verifies. It is rather surprising that the converse is often true; for example, the above Solecki's dichotomy results are proved for the $\leq_{RB}$-ordering (see also Lemma 2.1 below).

Any $\mathcal{I}$ serving as a counterexample to the Kechris–Mazur conjecture, or KMC, would have to be an F_σ P-ideal. Until recently, the only known F_σ P-ideals were the ***summable ideals***, that is, ones of the form

$$\mathcal{I}_f=\{A:\nu_f(A)<\infty\}=\{A:\lim_n(\nu_f(A\setminus n))=0\}$$

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where $\nu_f(A) = \sum_{n\in A} f(n)$ for some $f\colon \mathbb{N} \to \mathbb{R}^+$. These ideals can not serve as a counterexample to the Kechris–Mazur's conjecture, since we have either $\mathcal{I}_f \leq^+_{RB}$ Fin or $\mathcal{I}_{1/n} \leq^+_{RB} \mathcal{I}_f$ (note that this is false for the $\leq_{RB}$ ordering, and this is why we introduce $\leq^+_{RB}$). By [17, Theorem 3.3] (see also [11, Lemma 1.2]), all F_σ P-ideals are of the form

$$\mathcal{I} = \{A : \phi(A) < \infty\} = \{A : \lim_i \phi(A \setminus i) = 0\}$$

for some *lower semicontinuous* submeasure ϕ, i.e. mapping such that $\phi(A) \leq \phi(A \cup B) \leq \phi(A) + \phi(B)$, $\phi(\emptyset) = 0$ and $\lim_i \phi_i(A \cap i) = \phi(A)$ for all A, B. The first non-summable F_σ P-ideals were discovered in [4] (see also [3]). All these ideals were of the form

$$\mathcal{I}_{\{\phi_n\}} = \{A : \sum_i \phi_i(A) < \infty\},$$

where for some sequence $\{n_i\}$ each ϕ_i is a submeasure on the interval $[n_i, n_{i+1})$. But such ideals satisfy the KMC since if $\lim_i \sup_j \phi_i(\{j\}) = 0$ (and this can be assumed without a loss of generality by going to a positive set) then there are s_i and m_i such that

$$\phi_k(s_i) \begin{cases} \approx 1/i, & k = m_i \\ = 0, & \text{otherwise}, \end{cases}$$

so that the map collapsing s_i to i witnesses $\mathcal{I}_{1/n} \leq^+_{RB} \mathcal{I}_{\{\phi_i\}}$. An F_σ P-ideal which is not of the form $\mathcal{I}_{\{\phi_i\}}$ was later found by Solecki ([16]), who has also proved that this ideal is of the form $\mathcal{I}_{\{\phi_i\}}$ when restricted to a positive set, so it is again $\geq^+_{RB} \mathcal{I}_{1/n}$. Another class of F_σ P-ideals, suggested by Kechris, are ideals of the form

$$\mathcal{I} = \{A : \sqrt[p]{\sum_i \phi_i(A))^p} < \infty\}$$

for a sequence of submeasures ϕ_i as before and $p > 1$, but these again do not serve as a counterexample to KMC, for the same reason as $\mathcal{I}_{\{\phi_i\}}$. (However, using methods and results of [7] it can be proved that the Borel-cardinalities of these quotients are different for different p's.)

The new F_σ P-ideal which we define here is extracted from *Tsirelson space*, an infinite-dimensional Banach spaces which does not contain a copy of c_0 or any ℓ_p (see [2]). The study of this space has played a prominent role in the recent striking developments in the theory of infinite-dimensional Banach spaces (see [6], [13, page 956]). It is likely that other Banach spaces will give rise to interesting examples of analytic P-ideals (see [5]).

After the completion of this paper, we have learned that our main result, Theorem 3.1, was independently proved by B. Velickovic ([19]).

The paper is organized as follows. In §2 we prove that $\mathcal{P}(\mathbb{N})/\mathcal{I}_{1/n} \leq_B \mathcal{P}(\mathbb{N})/\mathcal{I}$ is equivalent to $\mathcal{I}_{1/n} \leq^+_{RB} \mathcal{I}$. In §3 we introduce the ideals $\mathcal{T}_{fh}$. In §§4–6 various properties of these ideals are proved, and in §7 we conclude the proof that $\mathcal{P}(\mathbb{N})/\mathcal{T}_{fh}$ serves as a counterexample to the Kechris–Mazur conjecture.

A word on notation: If s, t are finite sets of integers and n is an integer, by $s < t$ we will denote the fact that $\max s < \min t$, and by $n < s$ ($n > s$) the fact that $n < \min s$ ($n > \max s$, respectively).

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2. The first reduction

A quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}$ has the smaller *Borel-cardinality* than the quotient $\mathcal{P}(\mathbb{N})/\mathcal{J}$ (in symbols $\mathcal{P}(\mathbb{N})/\mathcal{I} \leq_B \mathcal{P}(\mathbb{N})/\mathcal{J}$) if there is a Borel mapping $F\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ such that $X\Delta Y \in \mathcal{I}$ iff $F(X)\Delta F(Y) \in \mathcal{J}$.

In the following lemma it is proved that the KMC is equivalent to its apparently stronger version, appearing in [4], which states: For every analytic ideal $\mathcal{I}$ such that $\mathcal{I} \not\leq_{RB}$ Fin one of Fin $\times\, \emptyset$, $\emptyset \times$ Fin or $\mathcal{I}_{1/n}$ is $\leq_{RB} \mathcal{I} \restriction A$ for some $\mathcal{I}$-positive A. (It is well-known that $\mathcal{P}(\mathbb{N})/\mathcal{I} \leq_B \mathcal{P}(\mathbb{N})/\mathrm{Fin}$ is equivalent to $\mathcal{I} \leq_{RB}$ Fin.)

Lemma 2.1. If $\mathcal{J}$ is an analytic P-ideal such that $\mathcal{P}(\mathbb{N})/\mathcal{I}_{1/n} \leq_B \mathcal{P}(\mathbb{N})/\mathcal{J}$, then $\mathcal{I}_{1/n} \leq^+_{RB} \mathcal{J}$. Moreover, there are $w_1 < w_2 < w_3 < \dots$ in Fin such that the map collapsing w_i to i witnesses this.

Proof By [17], we can fix a lower semicontinuous submeasure ϕ such that $\mathcal{J} = \{A : \lim_i \phi(A \setminus i) = 0\}$. Let $F\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ be a Borel reduction. By a standard use of stabilizers, similar to one below, we can assume that F is continuous (see also [18]). Find integers $1 = a_1 < b_1 < a_2 < b_2 < \dots$, $s_i \subseteq [b_i, a_{i+1})$ and $k_i \in (b_i, a_{i+1})$ so that for all i, all $u, v \subseteq b_i$, and all $X, Y \subseteq [a_{i+1}, \infty)$:

1. $a_i > 2^i$, $b_i > 2a_i$,
2. $(F(u \cup s_i \cup X)\Delta F(u \cup s_i \cup Y)) \cap k_i = \emptyset$,
3. $\phi((F(u \cup s_i \cup X)\Delta F(v \cup s_i \cup X)) \setminus k_i) \leq 2^{-i}$.

The method for construction of these sequences is standard, dating back to [15] and [18]: Assume that a_i, b_i $(i \leq n)$ and s_j, k_j $(j \leq n-1)$ as above have been chosen, but there are no s_n, k_n and a_{n+1} satisfying the requirements. The condition 1 is easy to satisfy and since F is continuous, 2 will be satisfied for every choice of s_n, a large enough k_n and a large enough a_{n+1}. Therefore we can construct a sequence $b_n < t_1 < l_1 < t_2 < l_2 < \dots$ so that $l_i \in \mathbb{N}$, $t_i \in$ Fin, and for all i there are $u_i, v_i \subseteq a_n$ such that

$$\phi((F(u_i \cup \bigcup_{j=1}^{i} t_j \cup t_{i+1})\Delta F(u_i \cup \bigcup_{j=1}^{i} t_j \cup t_{i+1})) \setminus l_i) > 2^{-n}.$$

Pick u, v such that $\langle u, v\rangle = \langle u_i, v_i\rangle$ infinitely often. Then $F(u \cup \bigcup_i t_i)\Delta F(v \cup \bigcup_i t_i)$ is not in $\mathcal{J}$—a contradiction.

Assume a_n, b_n, s_n and k_n are chosen to satisfy the above conditions. By 1, $\nu_{1/n}([a_i, b_i]) \geq 1/n$ and there is $u_i \subseteq (a_i, b_i)$ such that $|\nu_{1/n}(u_i) - 1/i| \leq 2^{-i}$ for every i. Let $C = \bigcup_i s_i$ and define $F_1 : P(\bigcup_i [a_i, b_i)) \to \mathcal{P}(\mathbb{N})$ by

$$F_1(B) = F(B \cup C)\Delta F(C).$$

Then $F_1(\emptyset) = \emptyset$ and for $X, Y \subseteq \bigcup_i [a_i, b_i)$ we have $(X \cup C)\Delta(Y \cup C) = X\Delta Y \in \mathcal{I}_{1/n}$ iff $F_1(X)\Delta F_1(Y) = F(X \cup C)\Delta F(Y \cup C) \in \mathcal{J}$. Note that by 2–3 for all i, all $u, v \subseteq \bigcup_{i \leq m} [a_i, b_i)$, and all $X, Y \subseteq \bigcup_{i \geq m+1} [a_i, b_i)$ we have:

4. $(F_1(u \cup X)\Delta F_1(u \cup Y)) \cap k_m = \emptyset$,
5. $\phi((F_1(u \cup X)\Delta F_1(v \cup X)) \setminus k_m) \leq 2^{-m}$.

Let $w_i = F_1(u_i) \cap [k_{i-1}, k_i)$. Then a map collapsing w_i to i witnesses $\mathcal{I}_{1/n} \leq^+_{RB} \mathcal{J}$. This follows by the following computations (assume $m \in A$ and let $u_X = \bigcup_{i \in X} u_i$):

$$\begin{aligned} t_m &= (F_1(u_A)\Delta w_A) \cap [k_{m-1}, k_m) \\ &\subseteq (F_1(u_A)\Delta F_1(u_{A\setminus m})) \cup (F_1(u_{A\setminus m})\Delta F_1(u_m))) \cap [k_{m-1}, k_m) \end{aligned}$$

(since $w_A \cap [k_{m-1}, k_m) = w_m$)

$$\subseteq ((F_1(u_A)\Delta F_1(u_{A\setminus m}))\setminus a_m)\cup(F_1(u_{A\setminus m})\Delta F_1(u_m)))\cap k_m$$

(by 4. and 5., since $u_A\Delta u_{A\setminus m}\subseteq\bigcup_{i=1}^{m-1}[a_i,b_i)$ and $u_{A\setminus m}\Delta u_m\subseteq\bigcup_{i=m+1}^{\infty}[a_i,b_i)$)

$$= (F_1(u_A)\Delta F_1(u_{A\setminus m}))\setminus a_m,$$

and therefore $\phi(t_m)\leq 2^{-m+1}$, if $m\in A$. An analogous computation shows that $\phi(t_m)\leq 2^{-m+1}$ also in the case when $m\notin A$, and therefore we have

$$F_1(u_A)\Delta w_A\subseteq\bigcup_m t_m\cup[1,k_1)\in\mathcal{J}.$$

To prove the moreover part, that we can assume that $w_1<w_2\dots$, find $1<n_1<m_1<n_2<m_2<\dots$ so that $n_i>2^i$, $\nu_{1/n}([n_i,m_i])-1/i|\leq 2^{-i}$, and

$$\max\bigcup_{j\in[n_i,m_i]}w_j<\min\bigcup_{j\in[n_{i+1},m_{i+1}]}w_j.$$

Then sets $w_i'=\bigcup_{j\in[n_i,m_i]}w_j$ are as required. □

Let us digress a little and note that $\emptyset\times\mathrm{Fin}$ also shares the nice property of $\mathcal{I}_{1/n}$ from Lemma 2.1, as its proof readily shows.

Lemma 2.2. If $\mathcal{J}$ is an analytic P-ideal such that $\mathcal{P}(\mathbb{N}^2)/\emptyset\times\mathrm{Fin}\leq_B\mathcal{P}(\mathbb{N})/\mathcal{J}$, then $\emptyset\times\mathrm{Fin}\leq_{RB}^+\mathcal{J}\restriction A$ for some $A\in\mathcal{J}^+$. □

It would be interesting to find more ideals with this property shared by $\mathcal{I}_{1/n}$ and $\emptyset\times\mathrm{Fin}$, since it considerably simplifies some questions about the Borel-cardinality of their quotients. Let us note that a pathological $F_{\sigma\delta}$ P-ideal $\mathcal{J}$ constructed in [3, §6] does not have this property. Namely, by a result of M.R. Oliver ([14]), $E_{\mathcal{J}}$ is Borel-reducible to $E_{\mathcal{Z}_0}$, the equivalence relation induced by the density zero ideal. But the ideal $\mathcal{Z}_0$ is nonpathological (see [3]), and therefore by [3, Proposition 6.5] $\mathcal{J}\leq_{RB}^+\mathcal{Z}_0$ would imply that $\mathcal{J}$ is nonpathological as well.

3. Tsirelson submeasures and ideals

Assume that $\{x_n\}$ is an unconditional basic sequence in a Banach space X such that $\lim_n\|\sum_{i=1}^n x_i\|=\infty$. Then

$$\mathcal{J}=\{A:\|\sum_{n\in A}x_n\|<\infty\}$$

is an analytic P-ideal, which we call a *generalized summable ideal*. Many analytic P-ideals are of this form, and ideals $\mathcal{T}_{fh}$ defined below are obtained in this way from the *Tsirelson space*, a Banach space which does not include a copy of c_0 or any ℓ_p (see [2]).

For sets $A,B\subseteq\mathbb{N}$ by AB we shall often denote their intersection, $A\cap B$. Fix functions $f\colon\mathbb{N}\to\mathbb{R}$ and an increasing $h\colon\mathbb{N}\to\mathbb{N}$. A tuple

$$\langle k,E_1,\dots E_{m)}\rangle$$

is *h-admissible* if $k\in\mathbb{N}$, $E_i\in\mathrm{Fin}$ for all i, $k<E_1<E_2<\dots<E_{m)}$, and $m\leq h(k)$. We shall denote pairs $\langle k,E_1,\dots,E_m\rangle$ by $\langle k,\vec{E}\rangle$ and write $m=|\vec{E}|$, so that the necessary condition for the admissibility is $|\vec{E}|\leq h(k)$. Let $\mathcal{A}_h$ be the set

of all h-admissible tuples. Define a sequence of *Tsirelson submeasures* $\tau_n = \tau_{f,h,n}$ ($n \in \mathbb{N} \cup \{\infty\}$) as follows:

$$\tau_{f,h,0}(A) = \sup_{n\in A} f(n)$$

$$\tau^k_{f,h,n}(A) = \sup_{\langle k,\vec{E}\rangle\in\mathcal{A}_h} \sum_{i=1}^{|\vec{E}|} \tau_{f,h,n}(E_iA),$$

$$\tau_{f,h,n+1}(A) = \max\{\tau_{f,h,n}(A), \frac{1}{2}\sup_k \tau^k_{f,h,n}(A)\},$$

$$\tau_{f,h,\infty}(A) = \sup_n \tau_{f,h,n}(A) = \lim_n \tau_{f,h,n}(A).$$

We shall always omit ∞ in subscripts, so that $\tau_{f,h}$ stands for $\tau_{f,h,\infty}$ and $\tau^k_{f,h}$ stands for $\tau^k_{f,h,\infty}$. Similarly, when f, h are clear from the context we shall write τ_n instead of $\tau_{f,h,n}$, τ^k instead of $\tau^k_{f,h}$, and so on. The submeasure $\tau = \tau_{f,h}$ defines a *Tsirelson ideal*, $\mathcal{T}_{fh}$, on $\mathbb{N}$ by:

$$\mathcal{T}_{fh} = \mathrm{Exh}(\tau) = \{A : \lim_i \tau(A \setminus i) = 0\}.$$

We shall always assume that $f\colon \mathbb{N} \to \mathbb{R}^+$, $\lim_i f(i) = 0$, and h is strictly increasing, unless otherwise specified.

Theorem 3.1. Assume f, h are as above. Then $\mathcal{P}(\mathbb{N})/\mathcal{I}_{1/n} \not\leq_B \mathcal{P}(\mathbb{N})/\mathcal{T}_{fh}$ and $\mathcal{T}_{fh}$ is an F_σ P-ideal.

The proof of Theorem 3.1 occupies the rest of this note.

Lemma 3.2. Assume f, h are as above. Then

1. Either $\tau(A) = \sup_{i\in A} f(i)$ or $\tau(A) = \frac{1}{2}\sup_k \tau^k(A)$.
2. All τ_n and τ are lower semicontinuous submeasures.
3. $\tau(A) < \infty$ if and only if $\lim_n \tau(A \setminus n) = 0$.
4. $\tau_{m+1}(A) < \infty$ if and only if $\lim_n \tau_m(A \setminus n) = 0$.
5. $\mathcal{T}_{fh} = \{A : \tau(A) < \infty\}$ and therefore it is an F_σ P-ideal.

PROOF To prove 1, note that if $\tau(A) > \sup_{i\in A} f(i)$ then we have

$$\tau(A) = \sup_n \left(\frac{1}{2}\sup_k \tau^k_n(A)\right) = \frac{1}{2}\sup_k \tau^k(A).$$

Statement 2 is obvious from the definition. In 3 only the direct implication requires a proof. Assume that $\lim_n \tau(A \setminus n) \neq 0$; then we can find $\varepsilon > 0$ and finite sets $w_1 < w_2 < \dots$ included in A such that $\tau(w_n) \geq \varepsilon$ for all n. Fix $p \in \mathbb{N}$ and find $\bar{n}$ such that

$$\min(w_{\bar{n}}) > 2p.$$

Then $A_0 = \bigcup_{i=\bar{n}}^{\bar{n}+2p} w_i \subseteq A$ and by 1 we get

$$\tau(A_0) \geq \frac{1}{2}\sum_{i=\bar{n}}^{\bar{n}+2p} \tau(w_i) \geq p \cdot \varepsilon.$$

Since p was arbitrary, we have $\tau(A) = \infty$ as required.

The proof of 4 is analogous to that of 3, and 5 follows immediately from 3. $\square$

We shall concentrate on the ideal $\mathcal{T}_{fh}$, but we note that the ideals

$$\mathcal{T}_{f,h,n} = \mathrm{Exh}(\tau_n) = \{A : \lim_i \tau_n(A \setminus i) = 0\}, \qquad n \in \mathbb{N},$$

can turn out to be interesting in their own right. All these are P-ideals which are not F_σ (assuming they are proper ideals, of course), since a mapping witnessing $\emptyset \times \mathrm{Fin} \leq_{RB} \mathcal{T}_{f,h,n}$ can be easily obtained from an $(n-1)$-good sequence (see §6).

4. Properties of Tsirelson submeasures

In this section we compute several lemmas which will be used in proof of Theorem 3.1. First we give a transparent description of how τ_n is computed in Lemma 4.2 below. By $\mathbb{N}^{<\mathbb{N}}$ we denote the set of all finite sequences of integers, and consider it as a tree under the ordering of end-extension. A set $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is a *tree* if it is closed under taking initial segments of its elements. Note that the height of T is equal to the maximal length of its element. By $\langle\rangle$ we denote the empty sequence in $\mathbb{N}^{<\mathbb{N}}$, and $t\hat{\ }i$ is the sequence obtained by concatenating t with $\langle i\rangle$. An *end-node* of T is any $t \in T$ such that $t\hat{\ }i \notin T$ for all i. (Note that end-nodes of T do not necessarily belong to its top level.)

Definition 4.1. A family $\langle E_t : t \in T\rangle$ is an h-*tree* if $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is a finitely branching finite tree, sets E_t are finite, and for all $t \in T$, if $t\hat{\ }1, \dots, t\hat{\ }l$ are the immediate successors of t in T then

1. $E_{t\hat{\ }1} < E_{t\hat{\ }2} < \dots < E_{t\hat{\ }l}$ and $E_t = \bigcup_{i=1}^l E_{t\hat{\ }l}$,
2. $l \leq h(\min E_{t\hat{\ }1})$, i.e. $\langle \min E_{t\hat{\ }1}, E_{t\hat{\ }1}, \dots, E_{t\hat{\ }l}\rangle$ is h-admissible, and
3. if t is an end-node of T, then E_t is a singleton.

The *height* of $\langle E_t : t \in T\rangle$ is the height of T. Note that every $i \in E_{\langle\rangle}$ defines the unique branch,

$$B_i = \{t \in T : i \in E_t\},$$

of T. The *length*, $|B_i|$, of this branch is equal to the length of its last node. A function $g\colon \mathbb{N} \to \{0, 1, \dots, n, \infty\}$ is an (h,n)-*weight assignment* if there is an h-tree $\langle E_t : t \in T\rangle$ of height at most n such that

4. $E_{\langle\rangle} = \{i : g(i) \neq \infty\}$,
5. $g(i) = |B_i|$ for each $i \in E_{\langle\rangle}$.

Lemma 4.2. Assume that $h\colon \mathbb{N} \to \mathbb{N}$ is strictly increasing. Then for every s we have

$$\tau_n(s) = \sup_g \sum_{i \in s} 2^{-g(i)} f(i),$$

where the supremum is taken over all (h,n)-weight assignments g.

Proof Note that every branching of an h-tree corresponds to an application of a step in the recursive definition of τ_{n+1}, and that the nodes of height k come with the weight equal to 2^{-k} because of the k-fold multiplication with $1/2$. Conditions 3. and 5. correspond to $\tau_0(A) = \sup_{i \in A} f(i)$. Therefore, the Lemma is proved by a straightforward induction on n. □

Lemma 4.3. If f, h are as in Lemma 4.2, $n \in \mathbb{N}$ and s is finite, then there is an $s' \subseteq s$ such that $\tau_n(s') = \tau_n(s)$ and $\tau(s') \leq 3\tau_n(s')/2$.

Proof Since s is finite, the supremum appearing in Lemma 4.2 is attained for some (h,n)-weight assignment g; let $s' = \{i : g(i) \neq \infty\} \cap s$. It suffices to prove that $\tau_m(s') \leq 3\tau_n(s')/2$ for every m. Fix m, let g_m be some (h,m)-weight assignment, and let $X = \{i \in s' : g_m(i) > g(i)\}$ and $Y = \{i \in s' : g_m(i) \leq g(i)\}$. We claim that

(†) $\sum_{i \in Y} 2^{-g_m(i)} f(i) \leq \tau_n(s')$.

To verify this, by Lemma 4.2 it suffices to show that the map g' defined by

$$g'(i) = \begin{cases} g_m(i), & \text{if } g_m(i) \leq g(i), \\ \infty, & \text{if } g_m(i) > g(i) \text{ or } g_m(i) = \infty, \end{cases}$$

is an (h,n)-weight assignment. To see this, let $\langle E_t : t \in T\rangle$ be an h-tree of height m witnessing that g_m is an (h,m)-weight assignment. Then for $T' = \{t \in T : |t| \leq n\}$ the family $\langle E_t \cap \{i : g'(i) \neq \infty\} : t \in T'\rangle$ is an h-tree (this follows immediately from the definitions). Since g is an (h,n)-weight assignment, this tree is of height at most n and it witnesses that g'_m is an (h,n)-weight assignment.

Therefore (†) is true and since $X \subseteq s'$ we have

$$\begin{aligned}\tau_m(s') &= \sum_{i \in X} 2^{-g_m(i)} f(i) + \sum_{i \in Y} 2^{-g_m(i)} f(i) \\ &\leq \frac{1}{2} \sum_{i \in X} 2^{-g(i)} f(i) + \tau_n(s') \leq \frac{1}{2}\tau_n(s') + \tau_n(s').\end{aligned}$$

Since m was arbitrary and $\tau(s') = \sup_m \tau_m(s')$, this concludes the proof. □

Lemma 4.4. Assume $\lim_n f(n) = 0$ and $h\colon \mathbb{N} \to \mathbb{N}$ is strictly increasing. Then for all s and n we have $\nu_f(s) \geq 2^{n+1}(\tau_{n+1}(s) - \tau_n(s))$.

Proof Fix an $\varepsilon > 0$. Let g be an $(h,n+1)$-weight assignment such that

$$\sum_{i \in s} 2^{-g(i)} f(i) \geq \tau_{n+1}(s) + \varepsilon,$$

as given by Lemma 4.2. Let $s_0 = \{i \in s : g(i) \leq n\}$. Then by a weight assignment argument identical to that in the proof of Lemma 4.3 we have $\tau_n(s) \geq \tau_n(s_0) \geq \sum_{i \in s_0} 2^{-g(i)} f(i)$, and therefore

$$\begin{aligned}2^{-(n+1)}\nu_f(s) &\geq 2^{-(n+1)} \sum_{i \in s \setminus s_0} f(i) \\ &\geq \sum_{i \in s} 2^{-g(i)} f(i) - \sum_{i \in s_0} 2^{-g(i)} f(i) \geq \tau_{n+1}(s) - \tau_n(s) + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this concludes the proof. □

Recall that if $\{w_i\}$ is a sequence of sets and $A \subseteq \mathbb{N}$, then we write $w_A = \bigcup_{i \in A} w_i$. The following lemma will be very useful (recall the definition of τ_n^k from §3).

Lemma 4.5. Assume f, h are as in Lemma 4.2, $w_1 < w_2 < \dots$ is a sequence of finite sets, and $\delta > 0$. If for all i we have

1. $\tau_n(w_i) < \delta/2$ and
2. $\tau_{n-1}^{\max(w_i)}(w_{[i+1,\infty)}) < \delta$ (taking $\max(w_0) = 1$),

then for every $A \subseteq \bigcup_i w_i$ we have $\tau_n(A) < \delta$.

Proof If $\tau_n(A) = \tau_{n-1}(A)$, the conclusion follows from 2. above.

Claim 4.6. Under the above assumptions, if $A \subseteq \bigcup_i w_i$ and $\tau_n(A) > \tau_{n-1}(A)$, then

$$\tau_n(A) \leq \sup_i \left(\tau_n(w_i) + \frac{1}{2}\tau_{n-1}^{\max w_i} \left(\bigcup_{j=i+1}^{\infty} w_j \right) \right).$$

PROOF Fix $\varepsilon > 0$. By the assumption, we have $\tau_n(A) \leq \frac{1}{2}\sum_{j=1}^{m} \tau_{n-1}(E_j A) + \varepsilon$ for some $m \leq h(k)$ and $k < E_1 < \cdots < E_m$. Let i be the minimal such that $k \leq \max w_i$. If $\max E_l \leq \max w_i$ for all $l = 1, \dots, h(k)$, then $\tau_n(A) \leq \tau_n(w_i) + \varepsilon$, and there is nothing to prove. Let l be the minimal such that $\max E_l > \max w_i$. Then

$$\begin{aligned}
\tau_n(A) &\leq \frac{1}{2}\sum_{j=1}^{l-1} \tau_{n-1}(E_j A) + \tau_{n-1}(E_l A) + \frac{1}{2}\sum_{j=l}^{m} \tau_{n-1}(E_j A) + \varepsilon \\
&\leq \frac{1}{2}\sum_{j=1}^{l-1} \tau_{n-1}(E_j A) + \tau_{n-1}(E_l(A w_i)) + \\
&\qquad \tau_{n-1}(E_l(A \setminus w_i)) + \frac{1}{2}\sum_{j=l}^{m} \tau_{n-1}(E_j A) + \varepsilon \\
&\leq \tau_n(w_i) + \frac{1}{2}\tau_{n-1}^{k}(\textstyle\bigcup_{p=i+1}^{\infty} w_p) + \varepsilon.
\end{aligned}$$

Since $k \leq \max w_i$ and $\varepsilon > 0$ was arbitrarily small, this completes the proof. □

Lemma 4.5 follows immediately by the Claim. □

5. The second reduction

The main result of this section is Proposition 5.3, essentially saying that if $\mathcal{I} \leq_{RB} \mathcal{T}_{fh}$ then $\mathcal{I}$ is of the form $\mathcal{T}_{f',h'}$ for some f', h' (possibly with $\lim_i f'(i) \neq 0$). It is essentially due to Casazza, Johnson and Tzafriri (see [1] and [2, Proposition I.12 and Lemmas II.1 and II.3]) who used the case when h is the identity function to prove that every infinite-dimensional subspace of the Tsirelson space includes a copy of a Tsirelson space. We reproduce the proof from [1] for the convenience of the reader.

Lemma 5.1. Assume $f \colon \mathbb{N} \to \mathbb{R}$ is nonnegative and $h \colon \mathbb{N} \to \mathbb{N}$ is strictly increasing, and let $h^+(n) = h(n + h(n))$. Then for every A and n we have

$$\tau_{f,h,n}(A) \leq \tau_{f,h^+,n}(A) \leq 3\tau_{f,h,n}(A).$$

PROOF The left-hand side inequality is obvious, since $h^+ \geq h$. We prove the following strengthening of the right-hand side inequality by induction:

(*) For all A and n there are sets $F_1 < F_2 < F_3$ such that

$$\tau_{f,h^+,n}(A) \leq \sum_{j=1}^{3} \tau_{f,h,n}(F_j A).$$

The case when $n = 0$ is trivial, so let us assume the lemma is proved for some n and prove it for $n+1$. If $\tau_{f,h^+,n+1}(A) = \tau_{f,h^+,n}(A)$, then there is nothing to prove, so we can assume

$$\begin{aligned}\tau_{f,h^+,n+1}(A) &= \frac{1}{2}\sum_{l=1}^{h^+(k)} \tau_{f,h^+,n}(E_l A), \qquad \text{for some } h^+\text{-admissible } \langle k, \vec{E}\rangle,\\ &\leq \frac{1}{2}\sum_{l=1}^{h^+(k)}\sum_{j=1}^{3} \tau_{f,h,n}(F_{lj}E_l A), \qquad \text{for some } F_{l1} < F_{l2} < F_{l3}.\end{aligned}$$

We can assume $F_{lj}E_l = F_{lj}$ for all l, j. For $G_{3(l-1)+j} = F_{lj}$ we have $G_1 < G_2 < \dots < G_{3h^+(k)}$. We can assume all G_l's are nonempty, possibly by eliminating the empty ones from the sequence. Let $k^* = k + h(k)$ (so that $h^+(k) = h(k^*)$) and

$$F_1 = \bigcup_{l=1}^{h(k)} G_l, \qquad F_2 = \bigcup_{l=h(k)+1}^{h(k)+h(k^*)} G_l, \quad \text{and} \quad F_3 = \bigcup_{l=h(k)+h(k^*)+1}^{3h^+(k)} G_l.$$

Then $\langle k, G_1, \dots, G_{h(k)}\rangle$ is h-admissible, so we have

$$\tau_{f,h,n+1}(F_1 A) \geq \frac{1}{2}\sum_{l=1}^{h(k)} \tau_{f,h,n}(G_l A).$$

Note that, since each G_l is nonempty, we have $\min(G_{h(k)+1}) \geq \min G_1 + h(k) \geq k^*$, and therefore the tuple associated with F_2 is h-admissible and we have

$$\tau_{f,h,n+1}(F_2 A) \geq \frac{1}{2}\sum_{l=h(k)+1}^{h(k^*)} \tau_{f,h,n}(G_l A).$$

Like before, $\min G_{h(k)+h(k^*)+1} \geq k^* + h(k^*)$. Since $k \leq h(k)$ and $h(2k) \leq h(k^*)$, we have (note that $h(i) + j \leq h(i+j)$, since h is strictly increasing):

$$3h^+(k) \leq h(k^*) + 2h(k^*) \leq h(k^*) + h(k^* + h(k^*))$$

and $3h^+(k) - h(k^*) \leq h(k^* + h(k^*))$, so the tuple associated with F_3 is h-admissible, and we have

$$\tau_{f,h^+,n+1}(A) \leq \frac{1}{2}\sum_{l=1}^{3h(2k))} \tau_{f,h,n}(G_l A) \leq \sum_{j=1}^{3} \tau_{f,h,n+1}(F_j A),$$

completing the inductive proof. □

Lemma 5.2. Assume f, h are as in Lemma 5.1 and that $w_1 < w_2 < \dots$ are finite sets. Let $f'(i) = \tau_{f,h}(w_i)$, $h'(i) = h(\min w_i)$, and $h''(i) = h(\max w_i)$. Then for all $A \subseteq \mathbb{N}$ we have

$$\tau_{f',h'}(A) \leq \tau_{f,h}(w_A) \leq 6\tau_{f',h''}(A).$$

Proof We first prove the left-hand side inequality, by proving

(*) $\tau_{f',h',n}(A) \leq \tau_{f,h}(w_A)$

using the induction on n. Since $\tau_{f',h'}(A) = \lim_n \tau_{f',h',n}(A)$, this will suffice. In the case when $n = 0$ for some $i \in A$ we have $\tau_{f',h',0}(A) = \tau_{f,h}(w_i) \leq \tau_{f,h}(\bigcup_{j\in A} w_j)$.

Now assume (*) is true for n. If $\tau_{f',h',n+1}(A)=\tau_{f',h',n}(A)$, there is nothing to prove. So we can assume

$$\tau_{f',h',n+1}(A)=\frac{1}{2}\sum_{j=1}^{h'(k)}\tau_{f',h',n}(E_jA),\qquad \text{for some } h\text{-admissible } \langle k,\vec{E}\rangle.$$

(Assuming that $|\vec{E}|=h'(k)$ is clearly not a loss of generality.) Since h' is increasing, we can assume $k=\min E_1$, and therefore $h'(\min E_1)=h(\min w_k))$. Let $E'_j=\bigcup_{j\in E_i}w_j$. Then $\langle \min w_k, E'_1,\dots,E'_{h'(k)}\rangle$ is h-admissible, by the inductive assumption we have

$$\frac{1}{2}\sum_{j=1}^{h'(k)}\tau_{f',h',n}(E_jA)\le\frac{1}{2}\sum_{j=1}^{h(\min w(k))}\tau_{f,h}(E'_jw_A)\le\tau_{f,h}(w_A),$$

and this ends the verification of the left-hand side inequality.

Now we prove the right-hand side inequality. Let $h^*(i)=2h''(i)$. Since $2h''(i)\le h''(i+h''(i))$, Lemma 5.1 implies $\tau_{f',h^*,n}\le 3\tau_{f'h'',n}$ for all n, and therefore it suffices to prove

$$\tau_{f,h,n}(w_A)\le 2\cdot\tau_{f',h^*,n}(A)$$

using the induction on n. When $n=0$ for some $i\in A$ we have

$$\tau_{f,h,0}(w_A)=\tau_{f,h,0}(w_i)\le\tau_{f,h}(w_i)=\tau_{f',h'',0}(i)\le 2\cdot\tau_{f',h^*,0}(A).$$

Now we assume the lemma is true for n and prove it for $n+1$. Again we can assume that $\tau_{f,h,n+1}(w_A)>\tau_{f,h,n}(w_A)$, therefore for some h-admissible $\langle k,\vec{E}\rangle$ (without a loss of generality, we can assume that $k=\min E_1$ and $|\vec{E}|=h(k)$) we have

$$\tau_{f,h,n+1}(w_A)=\frac{1}{2}\sum_{l=1}^{h(k)}\tau_{f,h,n}(E_lw_A).$$

We can assume $w_A\subseteq\bigcup_{l=1}^{h(k)}E_l$. Let $E_l^-=\{i:\min(w_i)\in E_l\}$ and $E_l^+=\{i:\max(w_i)\in E_l\}$. Then by the inductive assumption

$$\begin{aligned}\frac{1}{2}\sum_{l=1}^{h(k)}\tau_{f,h,n}(E_lw_A)&\le\frac{1}{2}\sum_{l=1}^{h(k)}2\tau_{f,h,n}(E_l^+A)+\frac{1}{2}\sum_{l=1}^{h(k)}2\tau_{f,h,n}(E_l^-A)\\&\le\frac{1}{2}\sum_{l=1}^{h(k)}2\tau_{f',h^*,n}(E_l^+A)+\frac{1}{2}\sum_{l=1}^{h(k)}2\tau_{f',h^*,n}(E_l^-A)\end{aligned}$$

If $j=\min(\bigcup_{l=1}^{h(k)}(E_l^+\cup E_l^-))$, then $h''(j)=h(\max w_j)\ge h(\min E_1)=h(k)$, also $h^*(j)=2h''(j)\ge 2h(k)$, so that $\langle j,\vec{E}^*\rangle$ (where $\vec{E}^*$ is the increasing enumeration of $\bigcup_{l=1}^{h(k)}(E_l^-\cup E_l^+)$) is h^*-admissible, so that the right-hand side is equal to at most $2\tau_{f',h^*,n+1}(A)$. As pointed out earlier, this concludes the proof since $\tau_{f',h^*,n+1}\le\tau_{f',h'',n+1}$ for all n. □

We are now prepared for the main result of this section.

Proposition 5.3. Assume $f\colon\mathbb{N}\to\mathbb{R}$ is nonnegative $h\colon\mathbb{N}\to\mathbb{N}$ is strictly increasing. If $w_1<w_2<\dots$ are finite sets, then for $f'(i)=\tau_{f,h}(w_i)$ and $h'(i)=h(\min w_i)$ we have $\mathcal{T}_{f',h'}=\{A:w_A\in\mathcal{T}_{fh}\}$.

PROOF Let $h''(i) = \max(w_i)$ and $(h')^+(i) = h'(i + h'(i))$. Since $h'' \leq (h')^+$, Lemmas 5.1 and 5.2 imply that for every A we have

$$\tau_{f'h'}(A) \leq \tau_{f,h}(w_A) \leq 6\tau_{f'h''}(A) \leq 6\tau_{f'(h')^+}(A) \leq 18\tau_{f'h'}(A),$$

thus $\tau_{f,h}(w_A) = \infty$ if and only if $\tau_{f'h'}(A) = \infty$, and the two ideals coincide. □

6. Good sequences

An important part of the proof of Theorem 3.1 is in proving its weaker version:

Proposition 6.1. If h is strictly increasing, the ideals $\mathcal{T}_{fh}$ and $\mathcal{I}_{1/n}$ are different.

PROOF Assume the contrary, that $\mathcal{T}_{fh} = \mathcal{I}_{1/n}$. Note that we can assume $\lim_i f(i) = 0$, for otherwise there would be an infinite set A none of whose infinite subsets is in $\mathcal{T}_{fh}$, but there is no such set for $\mathcal{I}_{1/n}$.

Claim 6.2. There is a sequence $t_1 < t_2 < \dots$ and $N \in \mathbb{N}$ such that

a) $\nu_f \leq N\nu_{1/n}$ on $\bigcup_i t_i$, and
b) $\inf_i(\tau_1(t_i)) > 0$.

PROOF Let $I_m = [m, m+h(m))$ and note that $\tau(I_m) = \tau_1(I_m)/2 = \nu_f(I_m)/2$. We shall prove that there is an $N \in \mathbb{N}$ such that if $t_m^0 = \{k \in I_m : k \cdot f(k) < N\}$, then all but finitely many t_m^0 satisfy $\nu_{1/n}(t_m^0) \geq (\ln 2)/2$. If can find such a sequence, then we can take t_i to be its subsequence satisfying $t_1 < t_2 < \dots$, and a) will be satisfied. To assure b), note that $\liminf_i \nu_f(t_i) > 0$, since otherwise there would be an infinite $C \subseteq \mathbb{N}$ such that $Y = \bigcup_{i \in C} t_i$ satisfies $\nu_f(Y) < \infty$, but then $Y \in \mathcal{T}_{fh} \setminus \mathcal{I}_{1/n}$, contradicting our assumptions. Therefore we can find a subsequence of t_i which satisfies $\inf_i \nu_f(t_i) > 0$, and since $t_i \subseteq [m_i, m_i + h(m_i))$ for some m_i, this implies b).

Assume that N as above does not exist. Then we can find $\{m(N)\}_{N=1}^{\infty}$ satisfying

1. $\nu_{1/n}\{k \in I_{m(N)} : k \cdot f(k) < 2^N\} < (\ln 2)/2$,
2. $k > 2^N$ for all $k \in I_{m(N)}$, $f(k) < 2^{-N}$, and
3. intervals $I_{m(N)}$ are pairwise disjoint.

Since $\nu_{1/n}(I_m) \geq \ln(m + h(m)) - \ln(m) \geq \ln 2$ for all m, by 1. we have

$$\nu_{1/n}\{k \in I_{m(N)} : k \cdot f(k) \geq 2^N\} > \frac{\ln 2}{2}.$$

Let $s_N \subseteq I_{m(N)}$ be such that

4. $k \cdot f(k) \geq 2^N$ for all $k \in s_N$, and
5. $\left|\nu_{1/n}(s_N) - 2^{-N+1}\right| < 2^{-N}$.

(Note that 5. can be assured by using 2.) Then we have

$$\tau(s_N) = \frac{\nu_f(s_N)}{2} \geq 2^N \nu_{1/n}(s_N) \geq 2^N \cdot 2^{-N} = 1$$

therefore $\bigcup_N s_N \in \mathcal{I}_{1/n} \setminus \mathcal{T}_{fh}$, contradicting our assumption and completing the proof. □

We will use $\{t_i\}$ given by Claim 6.2 to find $\{w_j\}$ such that for all j

6. $w_1 < w_2 < \dots$ are included in $\bigcup_i t_i$,
7. $\tau_{j+1}(w_j) \geq 2^{-j+1}$, and
8. $\tau_j(w_j) \leq 2^{-j}$.

Assume $\{w_j\}$ satisfy 6–8 above, and find $v_j \subseteq w_j$ such that $\tau(v_j) \leq 3\tau_{j+1}(w_j)/2$ and $\tau_{j+1}(v_j) = \tau_{j+1}(w_j)$. Therefore $\tau(v_j) < 2^{-j+2}$ and by Lemma 4.4 we have $\nu_f(v_j) \geq 2^{j+1}(2^{-j+1} - 2^{-j}) = 2$. Note that $\tau \leq \nu_f$ and $\nu_f \leq N\nu_{1/n}$ on the set $X = \bigcup_j v_j \subseteq \bigcup t_i$, and therefore $\lim_n \tau(X \setminus n) < \infty$ and $\lim_n \nu_f(X \setminus n) = \infty$. This implies $X \in \mathcal{T}_{fh} \setminus \mathcal{I}_f \subseteq \mathcal{T}_{fh} \setminus \mathcal{I}_{1/n}$, contradicting our assumptions.

Therefore it will suffice to find $\{w_i\}$ satisfying 6–8 above. We shall do this by using the following notion (recall that $u_A = \bigcup_{i \in A} u_i$).

Definition 6.3. A sequence $u_1 < u_2 < \dots$ of finite sets is *m-good for f, h* (or simply *m-good* if f, h are clear from the context) if there exists a $\delta > 0$ such that:

i) the set $\{\tau_m(u_i) : i \in \mathbb{N}\}$ is dense in $[0, \delta]$, and
ii) $\lim_i \tau_{m-1}(u_{[i,\infty)}) = 0$.

Lemma 6.4. Assume $\lim_n f(n) = 0$ and h is strictly increasing. If a sequence $\{u_i\}$ is m-good and $a, \varepsilon > 0$, then there is a finite $v \subseteq \bigcup_i u_i$ satisfying $|\tau_{m+1}(v) - a| < \varepsilon$ and $\tau_m(v) < \varepsilon$.

PROOF We can assume that $\varepsilon < \delta$, where δ is as in the definition of good sequence. By going to a subsequence we can also assume that for all i we have

(*) $\tau_{m-1}^{\max u_i}(u_{[i+1,\infty)}) < \varepsilon$ (taking $\max(u_0) = 1$).

The set $A = \{i : \tau_m(u_i) < \varepsilon/2\}$ is infinite and $\limsup_{k \in A} \tau_m(u_k) = \varepsilon/2$ (because the sequence $\{u_i\}$ is m-good and the corresponding δ is bigger than $\varepsilon/2$). By 4. of Lemma 3.2 we have $\tau_{m+1}(u_A) = \infty$. Therefore we can find $v \subseteq u_A$ such that $\tau_{m+1}(v) \geq a$, $\tau_{m+1}(v') < a$ for every $v' \subsetneq v$, and $f(j) < \varepsilon$ for every $j \in v$. Then by the subadditivity of τ_{m+1} we have $|\tau_{m+1}(v) - a| = \tau_{m+1}(v) - a < \varepsilon$. By Lemma 4.5 (applied with $\delta = \varepsilon$) we have $\tau_m(v) < \varepsilon$, therefore v is as required. □

Lemma 6.5. Assume $\lim_n f(n) = 0$ and h is strictly increasing. Then for every m-good sequence $\{u_i\}$ there is an $m+1$-good sequence $\{v_i\}$ such that $\bigcup_i v_i \subseteq \bigcup_i u_i$.

PROOF Let q_i ($i \in \mathbb{N}$) be an enumeration of all rationals in the interval $(0, 1)$. By using Lemma 6.4, we can recursively find $v_1 < v_2 < \dots$ included in $\bigcup_i u_i$ and such that $|\tau_{m+1}(v_i) - q_i| < 2^{-i}$ and $\tau_m(v_i) < 2^{-i}$. Then for every i we have $\tau_m(v_{[i,\infty)}) < 2^{-i+1}$, therefore the sequence v_i is $m+1$-good. □

Let t_i be as in Claim 6.2, and let $\varepsilon = \inf_i \tau_1(t_i) > 0$. Since $\lim_n f(n) = 0$, we can find $u_i \subseteq t_i$ ($i \in \mathbb{N}$) so that $\{\tau_1(u_i) : i \in \mathbb{N}\}$ is dense in $[0, \varepsilon]$. Since the condition $\lim_i \tau_0(u_i) = 0$ reduces to our assumption that $\lim_k f(k) = 0$, this sequence is 1-good. Therefore the sequence w_i as in 6–8 can now be constructed recursively by using Lemma 6.5 and Lemma 6.4. As explained before, this implies that that two ideals differ and concludes the proof. □

The above proof gives the following proposition of independent interest (see the proof of [5, Proposition 3.6]).

Proposition 6.6. If $\lim_n f(n) = 0$, $h\colon \mathbb{N} \to \mathbb{N}$ is strictly increasing and

$$\liminf_n \nu_f([n, n + h(n)) > 0,$$

then for every $m \geq 1$ there is an m-good sequence. □

7. Proof of Theorem 3.1

The ideal $\mathcal{T}_{fh}$ is, by 5 of Lemma 3.2, an F_σ P-ideal. Therefore it remains to prove that $\mathcal{P}(\mathbb{N})/\mathcal{I}_{1/n} \not\leq_B \mathcal{P}(\mathbb{N})/\mathcal{T}_{fh}$. By Lemma 2.1, it will suffice to prove that there is no sequence of finite sets $w_1 < w_2 < \dots$ such that for all $A \subseteq \mathbb{N}$ we have

(*) $A \in \mathcal{I}_{1/n}$ if and only if $\bigcup_{i\in A} w_i \in \mathcal{T}_{fh}$.

Assume that such a sequence exists. By Proposition 5.3, for some strictly increasing h' and $f'(n) = \tau_{f,h}(w_n)$ the ideals $\mathcal{I}_{1/n}$ and $\mathcal{T}_{f',h'}$ coincide, but this contradicts Proposition 6.1 and completes the proof.

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