BASIS PROBLEM FOR TURBULENT ACTIONS

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Abstract. We use modified Tsirelson's spaces [3] to prove that there is no finite basis for turbulent Polish group actions. This answers a question of Hjorth and Kechris (see [10, page 338], [8, Question 10.5]).

1. Introduction

One of the trends in Descriptive Set Theory during the last decade or so concerns the study of classifiability of Borel equivalence relations. Particularly well-studied are the orbit equivalence relations of Polish group actions (see [7, 10, 11, 8]). An equivalence relation is considered to be *classifiable* if we can associate some "simple" invariants to its equivalence classes in an effective way. The most general formalization of this notion is the one in which these simple invariants are taken to be countable structures (or equivalently, the isomorphism classes of countable structures). The first example of a Borel equivalence relation arising from a Polish group action which is not classifiable in this way was given by H. Friedman ([6]), and later rediscovered by Hjorth (see [8, §3.4]). This was the coset relation of the *zero density ideal*, $\mathcal{Z}_0 = \{A \subseteq \mathbb{N} : \lim_n |A \cap n|/n = 0\}$, considered as a subgroup of $\langle \mathcal{P}(\mathbb{N}), \Delta \rangle$ (with its compact metric topology). Later on, Hjorth isolated the dynamical property of group actions called *turbulence* (see [8, 12, 11]), and proved that in the realm of Borel equivalence relations arising from continuous Polish group actions, "non-classifiable (by countable structures)" is essentially equivalent with "induced by turbulent action" ([8, Theorem 0.14]). By analogy with the linearity phenomena occurring in the initial segment of $\leq_B$ (see [15, 10]), it seemed plausible that there should be a small basis for turbulent orbit relations. If $\mathcal{E}$ is a class of Borel equivalence relations, then we say that its subclass $\mathcal{E}_0$ is a *basis* for $\mathcal{E}$ if for every $E \in \mathcal{E}$ there is some $F \in \mathcal{E}_0$ such that $F \leq_B E$. The following was stated in ([8, Question 10.5]).

Question 1.1 (Hjorth–Kechris)**.** Is there a finite basis for turbulent orbit equivalence relations induced by continuous Polish group actions on Polish spaces?

In [10, Conjecture 7] it was conjectured that the coset relation associated with $\mathcal{Z}_0$ and one associated with the *summable ideal*, $\mathcal{I}_{1/n} = \{A : \sum_{i \in A} 1/i < \infty\}$ form such a basis. By [9, Theorem 5.3], $E_{\mathcal{I}_{1/n}}$ is a minimal turbulent orbit relation, and it is not reducible to $E_{\mathcal{Z}_0}$, so this seemed to be the simplest possible situation at the time. Recently M. Oliver ([16]) refuted this conjecture by proving that the orbit relation associated with $\mathcal{Z}_0$ is very far from being minimal. It is still unknown whether there are any minimal turbulent actions below $\mathcal{Z}_0$.

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Theorem 1.2. Every basis for turbulent orbit equivalence relations induced by continuous Polish group actions on Polish spaces is of size of the continuum.

The reader can consult [18] for a discussion of the basis problem in various settings.

Notation 1. A topological space is *Polish* if it is separable and completely metrizable. If E, F are Borel equivalence relations on Polish spaces X and Y, respectively, then E is *Borel reducible* to F, $E \leq_B F$, if there is a Borel-measurable $g\colon X \to Y$ such that $x \, E \, y$ if and only if $g(x) \, F \, g(y)$, for all $x, y \in X$. If u, v are finite subsets of $\mathbb{N}$ then $u < v$ means $\max u < \min v$. If $w_1 < w_2 < \dots$ is a sequence of finite sets then let $w_A = \bigcup_{i \in A} w_i$. The ordering $\leq^{++}_{RB}$ on the ideals is defined by

$$\mathcal{I} \leq^{++}_{RB} \mathcal{J} \text{ if there are finite sets } w_1 < w_2 < \dots \text{ such that } \mathcal{I} = \{A : w_A \in \mathcal{J}\}.$$

(This ordering was introduced in [14], where it was denoted by $\leq^{+}_{RB}$; it differs from the ordering denoted by $\leq^{+}_{RB}$ in [5].) An ideal $\mathcal{I}$ on $\mathbb{N}$ is *Polishable* if there is a Polish topology which makes $\langle \mathcal{I}, \Delta \rangle$ into a topological group. It is a *P-ideal* if it is σ-directed under the ordering of the inclusion modulo finite. By a result of Solecki ([17]), these two notions coincide: an analytic ideal $\mathcal{I}$ is a P-ideal if and only if it is Polishable. Let us say that a Polishable ideal $\mathcal{I}$ is *turbulent* if the natural action of $\mathcal{I}$ on $\mathcal{P}(\mathbb{N})$ is turbulent (see [8]). A sufficient condition for turbulence is that $\mathcal{I}$ is *dense*, i.e. that every infinite set of integers has an infinite subset in $\mathcal{I}$ (see [13]). For sets $A, B \subseteq \mathbb{N}$ by AB we shall often denote their intersection, $A \cap B$.

A familiarity with [5] would be helpful, since the proofs of main results of this note and [5] are quite analogous. We shall use the so-called *α-Tsirelson spaces* (see [3, page 95]), the natural generalization of the *Tsirelson space*, an ∞-dimensional Banach space which does not embed c_0 or any ℓ_p ($p \geq 1$). We shall assume that the reader has a copy of [5] handy.

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2. α-Tsirelson ideals

Fix $f\colon \mathbb{N} \to \mathbb{R}$, an increasing $h\colon \mathbb{N} \to \mathbb{N}$, and an $\alpha \in (0,1)$. An n-tuple

$$\langle k, s_1, \dots s_{h(k)} \rangle$$

is *h-admissible* if $k \in \mathbb{N}$, $s_i \in \mathrm{Fin}$ for all i and $k < s_1 < s_2 < \dots < s_{h(k)}$. Let $\mathcal{A}_h$ be the set of all h-admissible n-tuples. Define a sequence of *α-Tsirelson submeasures* $\tau_{\alpha,n} = \tau_{f,h,\alpha,n}$ ($n \in \mathbb{N} \cup \{\infty\}$) as follows:

$$\begin{aligned}
\tau_{f,h,\alpha,0}(A) &= \sup_{n \in A} f(n) \\
\tau^k_{f,h,\alpha,n}(A) &= \sup_{\langle k, \vec{E} \rangle \in \mathcal{A}_h} \sum_{i=1}^{h(k)} \tau_{f,h,\alpha,n}(s_i A), \\
\tau_{f,h,\alpha,n+1}(A) &= \max\{\tau_{f,h,\alpha,n}(A), \alpha \sup_k \tau^k_{f,h,\alpha,n}(A)\}, \\
\tau_{f,h,\alpha,\infty}(A) &= \sup_n \tau_{f,h,\alpha,n}(A) = \lim_n \tau_{f,h,\alpha,n}(A).
\end{aligned}$$

We shall always omit ∞ in subscripts, so that $\tau_{f,h,\alpha}$ stands for $\tau_{f,h,\alpha,\infty}$ and $\tau^k_{f,h,\alpha}$ stands for $\tau^k_{f,h,\alpha,\infty}$. The submeasure $\tau_{f,h,\alpha}$ defines an *α-Tsirelson ideal*, $\mathcal{T}_{f,h,\alpha}$, on $\mathbb{N}$ by:

$$\mathcal{T}_{f,h,\alpha} = \mathrm{Exh}(\tau_{f,h,\alpha}) = \{A : \lim_i \tau_{f,h,\alpha}(A \setminus i) = 0\}.$$

Therefore, $\tau_{f,h,1/2,n}$ is equal to $\tau_{f,h,n}$, as defined in [5].

Proposition 2.1. Every α-Tsirelson ideal is an F_σ P-ideal. If $\lim_i f(i) = 0$ and $\mathcal{I}$ is proper (i.e. $\mathbb{N} \notin \mathcal{I}$), then $\mathcal{I}$ turbulent.

PROOF Identical to [5, Lemma 3.2.5]. □

3. Properties of α-Tsirelson submeasures

Let us reproduce a definition from [5], to where we refer the reader for more details and definitions.

Definition 3.1. A family $\langle E_t : t \in T\rangle$ is an *h-tree* if $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is a finitely branching finite tree, sets E_t are finite, and for all $t \in T$, if $t\hat{}1, \dots, t\hat{}l$ are the immediate successors of t in T then

1. $E_{t\hat{}1} < E_{t\hat{}2} < \cdots < E_{t\hat{}l}$ and $E_t = \bigcup_{i=1}^l E_{t\hat{}l}$,
2. $l \leq h(\min E_{t\hat{}1})$, i.e. $\langle \min E_{t\hat{}1}, E_{t\hat{}1}, \dots, E_{t\hat{}l}\rangle$ is h-admissible, and
3. if t is an end-node of T, then E_t is a singleton.

The *height* of $\langle E_t : t \in T\rangle$ is the height of T. Note that every $i \in E_{\langle\rangle}$ defines the unique branch,

$$B_i = \{t \in T : i \in E_t\},$$

of T. The *length*, $|B_i|$, of this branch is equal to the length of its last node. A function $g\colon \mathbb{N} \to \{0, 1, \dots, n, \infty\}$ is an *(h,n)-weight assignment* if there is an h-tree $\langle E_t : t \in T\rangle$ of height at most n such that

4. $E_{\langle\rangle} = \{i : g(i) \neq \infty\}$,
5. $g(i) = |B_i|$ for each $i \in E_{\langle\rangle}$.

If g is an (h,n)-weight assignment, let us write

$$\mathrm{supp}(g) = \{i : g(i) < \infty\}.$$

We shall suppress writing f, h in subscripts whenever they are clear from the context. The following lemma is proved in [5, Lemma 4.2] in the special case when $\alpha = 1/2$, but the proof of the general case is identical and therefore we omit it.

Lemma 3.2. Assume $h\colon \mathbb{N} \to \mathbb{N}$ is strictly increasing and $\alpha \in (0,1)$. Then for every $s \subseteq \mathbb{N}$ we have

$$\tau_{\alpha,n}(s) = \sup_g \sum_{i \in s} \alpha^{g(i)} f(i),$$

where the supremum is taken over all (h,n)-weight assignments g. □

The following is an obvious extension of [5, Lemma 4.3] with the identical proof.

Lemma 3.3. If $s \subseteq \mathrm{supp}(\bar{g})$ for some (h,n)-weight assignment $\bar{g}$, then $\tau_{\alpha,n}(s) \leq \tau_\alpha(s) \leq (1+\alpha)\tau_{\alpha,n}(s)$.

PROOF It will suffice to prove that $\tau_{\alpha,m}(s) \leq (1+\alpha)\tau_{\alpha,n}(s)$ for $m \geq n$. Since $\mathrm{supp}(\bar{g})$ is finite, so is s, and there is some g such that $\tau_{\alpha,m}(s) = \sum_{i\in s} \alpha^{g(i)} f(i)$. Let $s^- = \{i \in s : g(i) \leq \bar{g}(i)\}$ and $s^+ = \{i \in s : g(i) \geq \bar{g}(i)+1\}$. Then

$$\begin{aligned}\tau_{\alpha,m}(s) = \sum_{i\in s^-} \alpha^{g(i)} f(i) = \sum_{i\in s^-} \alpha^{g(i)} f(i) + \sum_{i\in s^+} \alpha^{g(i)} f(i)&\\ \leq \tau_{\alpha,n}(s) + \alpha \sum_{i\in s^+} \alpha^{\bar{g}(i)} f(i) \leq (1+\alpha)\tau_{\alpha,n}(s).&\end{aligned}$$

□

Recall that $\nu_f(s) = \sum_{i\in s} f(i)$.

Lemma 3.4. Assume h is strictly increasing. If a finite set s and $\varepsilon, a > 0$ satisfy

1. $\sup_g \nu_f(s \cap \mathrm{supp}(g)) \leq \varepsilon$, where g ranges over all (h,n)-weight assignments, and
2. $\sup_g \nu_f(s \cap \mathrm{supp}(g)) = a$, where g ranges over all $(h,n+1)$-weight assignments,

then for every $\alpha \in (0,1)$ we have $\alpha^{n+1} a \leq \tau_{\alpha,n+1}(s) \leq \alpha^{n+1} a + \varepsilon$.

PROOF Since s is finite, by Lemma 3.2 there is an $(h,n+1)$-weight assignment g such that (we can assume that $\mathrm{supp}(g) \subseteq s$)

$$\tau_{\alpha,n+1}(s) = \sum_{i\in s} \alpha^{g(i)} f(i) \geq \sum_{i\in \mathrm{supp}(g)} \alpha^{n+1} f(i) = \alpha^{n+1} a.$$

To prove the other inequality, we define the *n-truncation of g, g'*, by

$$g'(i) = \begin{cases} g(i), & \text{if } g(i) \leq n,\\ \infty, & \text{if } g(i) > n \text{ or } g(i) = \infty. \end{cases}$$

This is an (h,n)-weight assignment (since the corresponding truncation of an h-tree associated with g is an h-tree, see the proof of [5, Lemma 4.3]), and therefore $\nu_f(\mathrm{supp}(g')) \leq \varepsilon$, by our assumptions. Since $g(i) = n+1$ for all $i \in \mathrm{supp}(g) \setminus \mathrm{supp}(g')$, we have

$$\begin{aligned}\tau_{\alpha,n+1}(s) &= \sum_{i\in s} \alpha^{g(i)} f(i)\\ &= \sum_{i\in \mathrm{supp}(g)\setminus\mathrm{supp}(g')} \alpha^{n+1} f(i) + \sum_{i\in \mathrm{supp}(g')} \alpha^{g'(i)} f(i)\\ &\leq \alpha^{n+1}\nu_f(\mathrm{supp}(g)) + \nu_f(\mathrm{supp}(g'))\\ &\leq \alpha^{n+1} a + \varepsilon\end{aligned}$$

This completes the proof. □

The following is a more precise version of [5, Lemma 4.4].

Lemma 3.5. If $\tau_{n+1,\alpha}(s) \geq b$ and $\tau_{n,\alpha}(s) \leq a$ for some $a, b \geq 0$, then

1. $\nu_f(s \cap \mathrm{supp}(g)) \leq \alpha^{-n} a$, for all (h,n)-weight assignments g, and
2. $\nu_f(s \cap \mathrm{supp}(g)) \geq \alpha^{-n-1}(b-a)$, for all $(h,n+1)$-weight assignments g.

PROOF For (1), let g be an (h,n)-weight assignment. Then

$$a \geq \tau_{\alpha,n}(s) \geq \sum_{i\in s} \alpha^{g(i)} f(i) \geq \sum_{i\in s\cap\mathrm{supp}(g)} \alpha^n f(i) = \nu_f(a \cap \mathrm{supp}(g)).$$

To prove (2), note that we can assume $\tau_{\alpha,n+1}(s) = b$ without a loss of generality. Since s is finite, there is an $(h, n+1)$-weight assignment $\bar{g}$ such that

$$b = \tau_{\alpha,n+1}(s) = \sum_{i \in s} \alpha^{\bar{g}(i)} f(i).$$

Let $\bar{g}'$ be the n-truncation of g, as defined in the proof of Lemma 3.4. Then we have

$$\begin{aligned} b &= \sum_{i \in s \setminus \mathrm{supp}(\bar{g}')} \alpha^{n+1} f(i) + \sum_{i \in \mathrm{supp}(\bar{g}')} f(i) \\ &\leq \alpha^{n+1} \nu_f(\mathrm{supp}(\bar{g}) \cap s) + \tau_{\alpha,n}(s) \\ &\leq \alpha^{n+1} \nu_f(a \cap \mathrm{supp}(\bar{g}) + a \end{aligned}$$

and (2) follows immediately. □

Proposition 3.6 below (which is related to a result of Bellenot ([1], [3, Proposition IV.c.8]) is going to be important in the proof of our main technical result, Theorem 5.1. This Proposition, as well as its proof and purpose are closely related to results of [5, §6].

Proposition 3.6. Assume that $\lim_i f(i) = 0$, $h\colon \mathbb{N} \to \mathbb{N}$ is strictly increasing, and $\liminf_n \nu_f([n, n+h(n)) > 0$. Then for all $\gamma, a > 0$ and every $n \in \mathbb{N}$ there is a $u \subseteq \mathbb{N}$ such that

$$\alpha^n a \leq \tau_\alpha(u) \leq 3\alpha^n a$$

for all $\alpha \in (\gamma, 1)$.

Proof It clearly suffices to prove the statement assuming $n \geq 2$. By [5, Proposition 6.6], our assumptions imply that there is an $n-1$-good sequence $u_1 < u_2 < \dots$ of subsets of $\mathbb{N}$ (see [5, Definition 6.3]). Let δ and b be such that

1. $\delta \leq (\gamma/2)^n a$,
2. $2^n(b - 2\delta) = a$.

Since $\lim_i f(i) = 0$, we can assume that

3. $f(k) < \delta$ for all $k \in \bigcup_i u_i$.

By [5, Lemma 6.4], there is $v_0 \subseteq \bigcup_i u_i$ such that

4. $|\tau_{n,1/2}(v_0) - b| < \delta$, and
5. $\tau_{n-1,1/2}(v_0) < \delta$.

By Lemma 3.5, we have

6. $\nu_f(v_0 \cap \mathrm{supp}(g)) \leq 2^{n-1}\delta$, for every $(h, n-1)$-assignment g, and
7. $\nu_f(v_0 \cap \mathrm{supp}(\bar{g})) \geq 2^n(b - 2\delta) = a$, for some (h, n)-assignment $\bar{g}$.

Then $v = v_0 \cap \mathrm{supp}(\bar{g})$ still satisfies (4) and (5), and by using (1) we can shrink it so that

8. $a \leq \nu_f(v) \leq a + \delta$.

By Lemma 3.4, for every $\alpha \in (\gamma, 1)$ we have

$$\alpha^n a \leq \alpha^n \nu_f(v) \leq \tau_{\alpha,n}(v) \leq \alpha^n \nu_f(v) + 2^{n-1}\delta \leq \alpha^n(a + \delta) + 2^{n-1}\delta,$$

and by Lemma 3.3, (6), and (7) we have

$$\alpha^n a \leq \tau_\alpha(v) \leq 2(\alpha^n(a + \delta) + 2^{n-1}\delta) \leq 3\alpha^n a,$$

as required. □

4. Reductions between the ideals

The following result is essentially due to Casazza, Johnston and Tzafriri ([2]).

Proposition 4.1. Assume $h\colon \mathbb{N} \to \mathbb{N}$ is strictly increasing and $0 < \alpha < 1$. If $w_1 < w_2 < \dots$ are finite sets, then for $f'(i) = \tau_{f,h,\alpha}(w_i)$ and $h'(i) = h(\min w_i)$ we have $\mathcal{T}_{f',h',\alpha} = \{A : w_A \in \mathcal{T}_{f,h,\alpha}\}$.

Proof This was proved in [5, Proposition 5.3] in the case when $\alpha = 1/2$; the proof of the general case is identical. □

Corollary 4.2. If $\mathcal{I}$ is an α-Tsirelson ideal and $\mathcal{J} \leq_{RB}^{++} \mathcal{I}$, then $\mathcal{J}$ is also an α-Tsirelson ideal. □

Lemma 4.3. Assume $h\colon \mathbb{N} \to \mathbb{N}$ is strictly increasing and let $h^+(n) = h(n+h(n))$. Then for every A and n we have $\tau_{f,h,\alpha,n}(A) \leq \tau_{f,h^+,\alpha,n}(A) \leq 3\tau_{f,h,\alpha,n}(A)$.

Proof In the case when $\alpha = 1/2$ this is [5, Lemma 5.1], and the proof of the general case is identical. □

Lemma 4.4. If $h_1(i) \leq h_2(i+1) \leq h_1(i+2)$, $\lim_i f(i) = 0$, and $0 < \alpha < 1$, then $\mathcal{T}_{f,h_1,\alpha} = \mathcal{T}_{f,h_2,\alpha}$.

Proof Since we have $h_1(i) \leq h_2(i+1) \leq h_2^+(i)$ and $h_2(i) \leq h_1(i+1) \leq h_1^+(i)$, Lemma 4.3 implies $(1/3)\tau_{f,h_1,\alpha} \leq \tau_{f,h_2,\alpha} \leq 3\tau_{f,h_1,\alpha}$, and this implies that the ideals coincide. □

Proposition 4.5. If $\mathcal{I}, \mathcal{J}$ are turbulent ideals such that $E_{\mathcal{I}} \leq_B E_{\mathcal{J}}$, then there is a turbulent ideal $\mathcal{J}_1$ such that $\mathcal{J}_1 \leq_{RB}^{++} \mathcal{I}$ and $\mathcal{J}_1 \leq_{RB}^{++} \mathcal{J}$.

Proof This follows from the proof of [5, Lemma 2.1], but in the case of F_σ-ideals (the only case which we shall use here) we shall prove a stronger statement in Theorem 6.2 below. □

Corollary 4.6. If $\mathcal{J}$ is an α-Tsirelson ideal and $\mathcal{I}$ is a turbulent ideal such that $E_{\mathcal{I}} \leq_B E_{\mathcal{J}}$, then there is an α-Tsirelson ideal $\mathcal{J}_1 \leq_{RB}^{++} \mathcal{I}, \mathcal{J}$. If moreover $\mathcal{I}$ is turbulent, then we can choose $\mathcal{J}_1$ to be turbulent.

Proof Let $\mathcal{J}_1$ be as guaranteed by Proposition 4.5. Then $\mathcal{J}_1$ is an α-Tsirelson ideal by Corollary 4.2. □

5. α-Tsirelson $\neq$ β-Tsirelson

Theorem 5.1 below will play a role analogous to that of Proposition 6.1 in [5].

Theorem 5.1. If $\alpha \neq \beta$ and $\mathcal{I}$ (respectively, $\mathcal{J}$) is a turbulent α- (respectively, β-) Tsirelson ideal, then $\mathcal{I} \neq \mathcal{J}$.

Proof Let us assume the contrary, that $\mathcal{T}_{f_1,h_1,\alpha} = \mathcal{T}_{f_2,h_2,\beta}$ are turbulent ideals, namely that $\lim_i f_1(i) = \lim_i f_2(i) = 0$. It would be helpful if we could assume that $f_1 = f_2$ and $h_1 = h_2$, and we shall begin by proving that we can indeed do so.

Lemma 5.2. Under the above assumption, there are f, h such that $\mathcal{T}_{f,h,\alpha} = \mathcal{T}_{f,h,\beta}$ is a turbulent ideal.

Proof Find $w_1 < w_2 < w_3 < \dots$ so that for all i we have

9. $|\tau_{f_1,h_1,\alpha}(w_i) - 1/i| < 2^{-i}$,
10. $i \leq h_1(\min(w_i)) < h_2(\min(w_{i+1})) < h_1(\min(w_{i+1}))$,

and let

11. $f_3(i) = f_1(\min w_i)$, $h_3(i) = h_1(\min w_i)$,
12. $f_4(i) = f_2(\min w_i)$, $h_4(i) = h_2(\min w_i)$.

Then Proposition 4.1 and our assumptions imply that

13. $\mathcal{T}_{f_3,h_3,\alpha} = \{A : w_A \in \mathcal{T}_{f_1,h_1,\alpha}\}$,
14. $\mathcal{T}_{f_4,h_4,\beta} = \{A : w_A \in \mathcal{T}_{f_2,h_2,\beta}\}$,
15. $\mathcal{T}_{f_3,h_3,\alpha} = \mathcal{T}_{f_4,h_4,\beta}$,
16. $i \leq h_3(i) < h_4(i+1) < h_3(i+2)$.

Therefore, Lemma 4.4 implies that

17. $\mathcal{T}_{f_4,h_4,\beta} = \mathcal{T}_{f_4,h_3,\beta}$,

and we can assume that $f_1 = f_2$, so one half of Lemma 5.2 is taken care of. The remaining part of the proof is very similar to the proof of [5, Claim 6.2]. In the following claim "positive" means positive with respect to any of the ideals from (17).

Claim 1. There is a positive set A such that

$$0 < \liminf_{i \in A} \frac{f_3(i)}{f_4(i)} \leq \limsup_{i \in A} \frac{f_3(i)}{f_4(i)} < \infty.$$

PROOF For $m \in \mathbb{N}$ let $I_m = [m, m + h_3(m))$, and note that

18. $\tau_{f_3,h_3,\alpha}(I_m) \geq \alpha\nu_{f_3}(I_m) \geq \alpha \ln(2)$,
19. $\nu_{f_3}(s) \geq \tau_{f_3,h_3,\alpha}(s) \geq \alpha\nu_{f_3}(s)$ for all $s \subseteq I_m$,
20. $\nu_{f_4}(s) \geq \tau_{f_4,h_3,\beta}(s) \geq \beta\nu_{f_4}(s)$ for all $s \subseteq I_m$.

For each n define a partition $I_m = I^-_{m,n} \cup I^0_{m,n} \cup I^+_{m,n}$ by

21. $I^-_{m,n} = \{i \in I_m : f_3(i)/f_4(i) < 1/n\}$,
22. $I^0_{m,n} = \{i \in I_m : 1/n \leq f_3(i)/f_4(i) \leq n\}$,
23. $I^+_{m,n} = \{i \in I_m : n < f_3(i)/f_4(i)\}$.

Observe that for every m we have

24. $I_m = I^0_{m,n}$ for all large enough n.

We claim that

25. $\lim_{n\to\infty} \sup_m \tau_{f_3,h_3,\alpha}(I_m \setminus I^0_{m,n}) = 0$.

Assume this fails, and that this limit is equal to some $\varepsilon > 0$. Then we can recursively find increasing sequences $n(i)$, $m(i)$ ($i \in \mathbb{N}$) such that for all i:

26. $\tau_{f_3,h_3,\alpha}(I_{m(i)} \setminus I^0_{m(i),n(i)}) \geq \varepsilon/2$,
27. $f_3(j), f_4(j) < 1/n(i)^2$ for all $j \in I_{m(i)}$,
28. $I_{m(1)} < I_{m(2)} < I_{m(3)} < \dots$,

(note that in the last line we are using (24)). Assume that for infinitely many i we have

29. $\tau_{f_3,h_3,\alpha}(I^-_{m(i),n(i)}) \geq \varepsilon/4$.

Then $\varepsilon/2 \leq \nu_{f_3}(I^-_{m(i),n(i)}) < \nu_{f_4}(I^-_{m(i),n(i)})/n(i)^2$, therefore we can find $s_i \subseteq I^-_{m(i),n(i)}$ such that

30. $\nu_{f_4}(s_i) \geq \varepsilon/4$ and $\nu_{f_3}(s_i) < \varepsilon/(4n(i)^2)$,

but by (19) and (20) the union of these s_i is in $\mathcal{T}_{f_3,h_3,\alpha} \setminus \mathcal{T}_{f_4,h_3,\beta}$, a contradiction. Therefore (29) fails, and since $I_m = I^-_{m,n} \cup I^0_{m,n} \cup I^+_{m,n}$ we have

31. $\tau_{f_3,h_3,\alpha}(I^+_{m(i),n(i)}) \geq \varepsilon/4$ for infinitely many i,

but then in a similar manner we can find $s_i \subseteq I^+_{m(i),n(i)}$ such that their union is in $\mathcal{T}_{f_4,h_3,\beta} \setminus \mathcal{T}_{f_3,h_3,\alpha}$. Therefore (25) is true, and we can find $\bar{n}$ such that

32. $\tau_{f_3,h_3,\alpha}(I_m \setminus I^0_{m,\bar{n}}) < (\alpha \ln 2)/2$

for all but finitely many m. By (18), $\tau_{f_3,h_3,\alpha}(I^0_{m,\bar{n}}) \geq (\alpha \ln 2)/2$ for such an m, and therefore by (19) and (20)

33. $A = \bigcup_m I^0_{m,\bar{n}}$

is a positive set on which $1/\bar{n} \leq f_3(i)/f_4(i) \leq \bar{n}$. □

Let $A = \{k_1, k_2, \dots\}$ be an increasing enumeration of the set guaranteed by Claim 1 and let

34. $\mathcal{I} = \{B \subseteq \mathbb{N} : \{k_i : i \in B\} \in \mathcal{T}_{f_3,h_3,\alpha}\}$,
35. $\mathcal{J} = \{B \subseteq \mathbb{N} : \{k_i : i \in B\} \in \mathcal{T}_{f_4,h_3,\beta}\}$,
36. $f(i) = f_3(k_i)$, $f'(i) = f_4(k_i)$, and $h(i) = h_3(k_i)$.

Then clearly $\mathcal{I} = \mathcal{T}_{f,h,\alpha}$, $\mathcal{J} = \mathcal{T}_{f',h,\beta}$, and $\mathcal{I} = \mathcal{J}$. Moreover, since $1/\bar{n} \leq f/f' \leq \bar{n}$, we also have $\mathcal{T}_{f',h,\beta} = \mathcal{T}_{f,h,\beta}$, and therefore $\mathcal{T}_{f,h,\alpha} = \mathcal{T}_{f,h,\beta}$. These ideals are turbulent, since $\lim_i f(i) \leq \limsup_i f_3(i) = 0$; this completes the proof of Lemma. □

To complete the proof of Theorem, we need to show that the assumptions of Proposition 3.6 are satisfied, namely that we can assume

$$\liminf_m \nu([m, m + h(m)) > 0.$$

Let $w_1 < w_2 < \dots$ be sets such that for all i

37. $|\nu_f(w_i) - 1/i| < 2^{-i}$,
38. $h(\min(w_i)) \geq i$, for all i.

Then $\mathcal{I} = \{B : w_B \in \mathcal{T}_{f,h,\alpha}\} = \{B : w_B \in \mathcal{T}_{f,h,\beta}\}$ is a Tsirelson ideal such that this limit is $\geq \log(2)$.

Now fix any $0 < \gamma < \alpha, \beta$ and apply Proposition 3.6 with $a = 1/\beta^n$ and $\varepsilon = \alpha^n/\beta^n$ to pick u_n $(n \in \mathbb{N})$ such that

$$\left(\frac{\alpha}{\beta}\right)^n \leq \tau_{f,h,\alpha}(u_n) \leq 3\left(\frac{\alpha}{\beta}\right)^n \quad \text{and} \quad 1 \leq \tau_{f,h,\beta}(u_n) \leq 3.$$

Then $\bigcup_n u_n \in \mathcal{T}_{f,h,\alpha} \setminus \mathcal{T}_{f,h,\beta}$. □

6. Reductions below ideals

Recall that an equivalence relation E is *countable* if it all of its equivalence classes are countable, and that it is *essentially countable* if it is reducible to some countable equivalence relation. Turbulent orbit equivalence relations can not be essentially countable (since they do not admit classification by countable structures, by [8, Theorem 0.14]). Recall that $\mathcal{I}_{1/n} = \{A : \sum_{i \in A} 1/i < \infty\}$. The following theorem from [9, Theorem 5.3] shows that $E_{1/n}$ is a minimal Borel turbulent orbit equivalence relation.

Theorem 6.1 (Hjorth)**.** If E is a Borel equivalence relation reducible to $E_{\mathcal{I}_{1/n}}$, then one of the following applies:

1. E is essentially countable.
2. $E_{1/n} \leq_B E$. □

A minor change of the proof gives us the following result.

Theorem 6.2. If $\mathcal{I}$ is a turbulent F_σ P-ideal and $E \leq_B E_\mathcal{I}$ is a Borel equivalence relation, then one of the following applies:

1. E is essentially countable.
2. There is a turbulent ideal $\mathcal{J}$ such that $E_\mathcal{J} \leq_B E$ and $\mathcal{J} \leq_{RB}^{++} \mathcal{I}$.

Proof An analysis of the proof of Theorem 6.1 reveals that it can be split into two steps, the first one being the proof of Theorem 6.2 and the second one arguing that if $\mathcal{J} \leq_{RB}^{++} \mathcal{I}_{1/n}$ and $\mathcal{J}$ is turbulent, then $\mathcal{I}_{1/n} \leq_{RB}^{++} \mathcal{J}$. The only property of the summable ideal $\mathcal{I}_{1/n}$ used to prove Theorem 6.2 is that $\nu_{1/n}(A) = \sum_{n\in A} 1/n$ is a lower semicontinuous submeasure such that

$$\begin{aligned}\mathcal{I}_{1/n} &= \{A : \lim_n \nu_{1/n}(A \setminus [1,n)) = 0\}\\ &= \{A : \nu_{1/n}(A) < \infty\}.\end{aligned}$$

But by a result of Solecki ([17]) every F_σ P-ideal is of this form, when $\nu_{1/n}$ is replaced with some lower semicontinuous submeasure. □

Corollary 6.3. If $\mathcal{I}$ is an α-Tsirelson ideal and E is a Borel equivalence relation reducible to $E_\mathcal{I}$, then (exactly) one of the following applies:

3. There is an α-Tsirelson ideal $\mathcal{J}$ such that $E_\mathcal{J} \leq_B E$.
4. E is essentially countable.

Proof By Theorem 6.2, if E is not essentially countable then there is a turbulent ideal $\mathcal{J} \leq_{RB}^{+} \mathcal{I}$ such that $E_\mathcal{J} \leq_B E$. By Corollary 4.6, we can assume that $\mathcal{J}$ is an α-Tsirelson ideal. □

7. α-Tsirelson and β-Tsirelson are incompatible

Let us write $\mathcal{E}_\alpha$ for the class of all Borel equivalence relations induced by turbulent actions which are reducible to some α-Tsirelson equivalence relation. A class $\mathcal{E}$ of equivalence relations is *downwards directed* if for all $F_1, F_2 \in \mathcal{E}$ there is some $E \in \mathcal{E}$ which is reducible to both F_1 and F_2.

The following theorem is not being used in the proof of Theorem 1.2, but it is a natural complement to (and much easier to prove than) Theorem 7.2, which directly implies Theorem 1.2.

Theorem 7.1. Each $\mathcal{E}_\alpha$ $(0 < \alpha < 1)$ is downwards directed.

Proof Fix $F_1, F_2 \in \mathcal{E}_\alpha$. We need to find $E \in \mathcal{E}_\alpha$ which is $\leq_B F_1, F_2$. By Theorem 6.2 we can assume that both F_1 and F_2 are induced by ideals, say $\mathcal{T}_{f,h,\alpha}$ and $\mathcal{T}_{f',h',\alpha}$, such that $\lim_i f(i) = \lim_i f'(i) = 0$ (this latter condition is equivalent to turbulence, see [13]). Find subsets $u_1 < u_2 < \dots$ and $v_1 < v_2 < \dots$ of $\mathbb{N}$ such that

1. $|\tau_{f,h,\alpha}(u_i) - 1/i| < 2^{-i}$,
2. $|\tau_{f',h',\alpha}(v_i) - 1/i| < 2^{-i}$,
3. $h(\min u_i) < h'(\min v_i) < h(\min u_{i+1})$,
4. $f_1(i) = 1/i$, $h_1(i) = h(\min(w_i))$, $h_2(i) = h'(\min(v_i))$.

Since $\lim_i f_1(i) = \lim_i f_2(i) = 0$, this can be arranged easily. If u_i and v_i are as above, then

$$\mathcal{T}_{f_1,h_1,\alpha} = \{A : u_A \in \mathcal{T}_{f,h,\alpha}\},$$
$$\mathcal{T}_{f_1,h_2,\alpha} = \{A : v_A \in \mathcal{T}_{f,h,\beta}\},$$

and these two ideals are identical, by Lemma 4.4 and (3). This is a turbulent α-Tsirelson ideal, and $E = E_{\mathcal{I}}$ is reducible to both F_1 and F_2. □

The following result roughly corresponds to a well-known fact about α-Tsirelson spaces (see [3, Theorem X.a.3]).

Theorem 7.2. If $\alpha \neq \beta$, $F_1 \in \mathcal{E}_\alpha$, $F_2 \in \mathcal{E}_\beta$, and E is reducible to both F_1 and F_2, then E has to be essentially countable.

Proof Let $\mathcal{I}$ be a turbulent α-Tsirelson ideal, let $\mathcal{J}$ be a turbulent β-Tsirelson ideal. Assume that E is not essentially countable, and therefore not turbulent, yet $E \leq_B E_{\mathcal{I}}$ and $E \leq_B E_{\mathcal{J}}$. By Theorem 6.2 applied to E and $E_{\mathcal{I}}$, we obtain $E_{\mathcal{I}'} \leq_B E$ induced by some turbulent α-Tsirelson ideal $\mathcal{I}' \leq_{RB}^{++} \mathcal{I}$. Now applying Lemma 4.6 to $E_{\mathcal{I}'}$ and $E_{\mathcal{J}}$, we obtain a turbulent β-Tsirelson ideal $\mathcal{I}'' \leq_{RB}^{++} \mathcal{I}', \mathcal{J}$. Therefore, by Corollary 4.2, $\mathcal{I}''$ is both α-Tsirelson and β-Tsirelson, and an application of Theorem 5.1 gives $\alpha = \beta$ and therefore concludes the proof. □

Let us finally prove the result stated in the Introduction.

Proof [Theorem 1.2] Since there are only continuum many Borel equivalence relations, all turbulent orbit equivalence relations form a basis of size of the continuum. Therefore it suffices to prove that if E_ξ $(\xi < \lambda < \mathfrak{c})$ are equivalence relations induced by turbulent actions then there is some equivalence relation E induced by a turbulent action such that $E_\xi \not\leq_B E$ for all $\xi < \lambda$. But Theorem 7.2 implies that for every ξ there is at most one α_ξ such that E_ξ is reducible to some α_ξ-Tsirelson ideal. If $\alpha \in (0,1)$ is different from all α_ξ, then any equivalence relation E associated with some α-Tsirelson ideal is as required. □

8. Concluding remarks

As suggested by Hjorth, finding a turbulent orbit equivalence relation with no minimal one below it would witness the ultimate failure of any version of [10, Conjecture 7]. At present, the only known minimal turbulent orbit relation is $E_{\mathcal{I}_{1/n}}$ (by [9], see Theorem 6.1).

Conjecture 8.1. For every $\alpha \in (0,1)$, $\mathcal{E}_\alpha$ does not contain minimal turbulent orbit equivalence relations.

It should be noted that an analogous statement is indeed true in the context of Banach spaces: every α-Tsirelson space T has a complemented subspace $T' \subseteq T$ into which T does not embed (see [1, 3]).

Let us note that every Tsirelson submeasure is equal to the supremum of the measures it dominates, and therefore the results of [4] apply to show that one quotient Boolean algebra over a Tsirelson ideal is Borel-embeddable into another if and only if there is a $\leq_{RB}$-reduction between the corresponding ideals. This makes it relatively easy to confirm the following, weaker version of the above conjecture.

Proposition 8.2. For every turbulent α-Tsirelson ideal $\mathcal{I}$ there is a turbulent α-Tsirelson ideal $\mathcal{J}$ whose quotient Boolean algebra is Borel-embeddable into $\mathcal{P}(\mathbb{N})/\mathcal{I}$, but not vice versa. $\square$

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