FINITE RANK TFA GROUPS

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§1. A classical result

Definition A *TFA group* is an abelian torsion free ($g + g + ...g = 0 \Rightarrow g = 0$) group. It has *finite* rank n if any $n+1$ many elements are dependent – that is to say, for any $g_1, ..., g_n, g_{n+1}$ there are $k_1, ..., k_{n+1} \in \mathbb{Z} \setminus \{0\}$ with

$$k_1 \cdot g_1 + ...k_{n+1} \cdot g_{n+1} = 0.$$

Here

$$k \cdot g =_{df} g + g + ... \quad {}^{K \text{ times}} \quad + g.$$

The rank n TFA groups are exactly the subgroups of $(\mathbb{Q}^n, +)$.

Theorem (Baer) There is a "nice" classification of TFA_1 groups.

Sketch of Proof. Let $A \in \mathrm{TFA}_1$. For any $g \in A$ not equal to the identity let

$$\theta(A, g) = \{(p, n) : p^n \text{ divides } g\}.$$

Claim: If q is a prime, then
(i) $p \neq q$ implies

$$(p, n) \in \theta(A, g) \text{ iff } (p, n) \in \theta(A, q \cdot g);$$

(ii) $p = q$ implies

$$(p, n+1) \in \theta(A, g) \text{ iff } (p, n) \in \theta(A, q \cdot g).$$

($\square$ Claim)

Thus for any two non-zero $g, h \in A$ the symmetric difference $\theta(A, g)\Delta\theta(A, h)$ is finite.

Moreover, for any suitably "convex"

$$S \subset \mathrm{Primes}\times\mathbb{N}$$

$$|S\Delta\theta(A, g)| < \aleph_0$$

we may find $k \in A$ with

$$\theta(A, k) = X.$$

Conversely

Claim: If $A, B \in \mathrm{TFA}_1$,

$$g \in A, \;\; h \in B$$

$$\theta(A, g) = \theta(B, h)$$

then there is an isomorphism between A and B sending g to h.

($\square$ Claim)

Thus

$$A \mapsto \{\theta(A, g) : g \neq 0\}$$

provides a complete invariant for the isomorphism type of A, and in fact we can obtain elements of

$$\mathcal{P}(\mathrm{Primes} \times \mathbb{N})/\mathrm{Finite}$$

(subsets of Primes$\times$ natural numbers considered up to finite difference) as complete invariants for rank one torsion free abelian groups.

$\square$

1.3 Question Does a similarly nice classification exist for general finite rank TFA groups?

We need to make precise the meaning of “nice”. We need definability assumptions on the functions used in a classification theorem – since unbridled use of the axiom of choice would enable us to “classify” virtually anything by virtually anything else, subject only to agreement in cardinality.

Perhaps the most generous class of functions for which we can define and prove theorems without any metamathematical entanglement are the Borel functions.

1.4 Theorem In suitable Borel structures, the finite rank TFA groups do not admit a Borel assignment of complete invariants in

$$\mathcal{P}(\mathbb{N})/\text{Finite}.$$

Here $\mathcal{P}(\mathbb{N})$ (or $\mathcal{P}(S)$ for any countable set S) may be naturally identified with $2^{\mathbb{N}}$ – and we obtain a reasonable Borel structure from the product topology.

§2. Borel structure

2.1 Definition A *a Polish space* is a separable space allowing a complete metric compatible with its topology.

2.2 Examples $\mathbb{R}$. $\mathbb{C}$.

$2^{\mathbb{N}}$ in the product topology – for instance with

$$d(f,g) = 2^{-\mu n\{f(n)\neq g(n)\}}.$$

Any compact metric space.

Given any countable set S we may identify $\mathcal{P}(S)$ (the set of subsets of S) with 2^S and endow it with a Polish topology.

2.3 Fact Closed (and even G_δ) subsets of Polish spaces are again Polish in the subspace topology.

2.4 Definition Let TFA_n be the set $\{G \subset \mathbb{Q}^n : G$ is a subgroup of $(\mathbb{Q}^n, +)\}$.

TFA_n is a closed subspace of $\mathbb{Q}^n$, and hence Polish in its own right. Every rank n TFA abelian group can be realized as an element of TFA_n.

In this sense we can obtain a natural Borel structure on the rank n torsion free abelian groups.

§3 Equivalence relations

3.1 Definition If E and F are equivalence relations on Polish spaces X and Y then

$$E \leq_B F$$

indicates that there is a Borel function

$$\theta : X \to Y$$

such that for all $x_1, x_2 \in X$

$$x_1 E x_2 \Leftrightarrow \theta(x_1) F \theta(x_2).$$

A function is *Borel* if it pulls open sets back to Borel sets – that is, sets in the σ-algebra generated by the open sets.

3.2 Examples For X a Polish space, let $\mathrm{id}(X)$ be the identity equivalence relation on X.

Let E_0 be the equivalence relation on $2^{\mathbb{N}}$ of eventual agreement – $x_1 E_0 x_2$ if there is some N such that for all $n > N$

$$x_1(n) = x_2(n).$$

Similarly, for any countable set S we can define $E_0(S)$ on

$$\mathcal{P}(S) \sim 2^S$$

as the equivalence relation of agreeing on a cofinite set; for any countably infinite set S we have

$$E_0 \leq_B E_0(S) \leq_B E_0.$$

Let $\cong_n$ be the equivalence relation on isomorphism on TFA_n (the space of rank n TFA groups with underlying set $\mathbb{N}$).

Baer's theorem shows that

$$\cong_1 \leq_B E_0.$$

3.3 Definition An equivalence relation is *countable* if every equivalence class is countable; an equivalence relation E on X is *Borel* if it is Borel as a subset of $X \times X$.

Each $\cong_n$ (isomorphism on TFA_n) is Borel reducible to a countable equivalence relation – but it is not true that all countable Borel equivalence relations are Borel reducible to E_0.

3.4 Definition An equivalence relation E is *smooth* if $E \leq_B \mathrm{id}(\mathbb{R})$ – that is to say we may assign reals as complete invariants.

A countable equivalence relation is *hyperfinite* if we may in a Borel manner assign the structure of a $\mathbb{Z}$ chain to each infinite equivalence class.

A countable equivalence relation is *treeable* if we may in a Borel fashion assign the structure of a tree (connected graph with no cycles) to each equivalence relation.

A countable Borel equivalence relation is smooth if and only if there is a Borel set meeting each orbit in exactly one point.

A countable Borel equivalence relation is hyperfinite if and only if it is $\leq_B$ reducible to E_0.

Smooth $\subset$ hyperfinite $\subset$ treeable $\subset$ "treeable $\times$ treeable" $\subset$ countable.

None of these implications reverse.

3.5 Lemma If E and F are countable Borel equivalence relations on Polish X and Y, with $E \leq_B F$ then

(i) F smooth implies E smooth;

(ii) F hyperfinite implies E hyperfinite;

(iii) F treeable implies E treeable.

3.6 Lemma If E and F are countable Borel equivalence relations on Polish X and $E \subset F$

(i) F smooth implies E smooth;

(ii) F hyperfinite implies E hyperfinite;

(iii) F treeable implies E treeable.

§4 Results

If G_1, G_2 are subgroups of $\mathbb{Q}^n$ and $\pi : G_1 \cong G_2$ is an isomorphism, then π extends to an automorphism

$$\Pi : \mathbb{Q}^n \cong \mathbb{Q}^n.$$

Thus we may view $\cong_n$ (isomorphism) on rank n TFA groups as being induced by the automorphism group of $\mathbb{Q}^n$: $\mathrm{GL}(n, \mathbb{Q})$.

It is widely believed that:

4.1 Conjecture If G is an abelian (or even *amenable*) countable group acting by Borel automorphisms on a Polish space X with induced orbit equivalence relation E_G then

$$E_G \leq E_0.$$

(That is, E_G should be hyperfinite.)

Since $\mathrm{GL}(1, \mathbb{Q}) = \mathbb{Q}$ is responsible for $\cong_1$, it may be the case that Baer's classification theorem is a special case of this more general conjecture.

Two reasons for expecting the situation could be more complicated in higher dimensions.

4.2 Fact (Folklore)

(i) For $n \geq 2$, the groups $\mathrm{GL}(n, \mathbb{Q})$, $\mathrm{SL}(n, \mathbb{Z})$, $\mathrm{PSL}(n, \mathbb{Z}) = \mathrm{SL}(n, \mathbb{Z})/\{I, -I\}$ all fail to be amenable.

(ii) If G is a non-amenable group acting freely by measure preserving transformations on a *probability* space then the corresponding orbit equivalence relation E_G is not *hyperfinite*.

4.3 Theorem

(i) For $n \geq 3$ the group $\mathrm{SL}(n, \mathbb{Z})$ is *Kazhdan*.

(ii) (Adams, Spatzier) If a countable Kazhdan group G acts by measure preserving transformations on a probability space then the corresponding orbit equivalence relation E_G is not *treeable* whenever there is an infinite subgroup acting freely.

4.4 Theorem For $n \geq 3$ the isomorphism relation on rank n TFA groups, $\cong_n$ on TFA_n is not treeable.

Outline of proof. The main issue is to somehow find a measure.

Consider the dual group $\Gamma(\mathbb{Q}^n/\mathbb{Z}^n)$ of homomorphism

$$\psi : (\mathbb{Q}^n/\mathbb{Z}^n, +) \to \mathbb{T} = (\mathbb{R}/\mathbb{Z}, +).$$

Since $\mathbb{Q}^n/\mathbb{Z}^n$ is countable, the dual group $\Gamma(\mathbb{Q}^n/\mathbb{Z}^n)$ is a compact metric group, and so there is an invariant probability (Haar) measure, μ.

Let $\pi_n : \mathbb{Q}^n \twoheadrightarrow \mathbb{Q}^n/\mathbb{Z}^n$ be the canonical projection. For $\psi \in \Gamma$, let

$$K(\psi) = \{\vec{q} \in \mathbb{Q}^n : \psi(\pi_n(\vec{q})) = 0\}.$$

Then push the Haar measure μ on the dual group along the "kernel" map K to obtain $\nu = K_*(\mu)$ –

$$\nu(B) = \mu(K^{-1}[B]).$$

Here one can show that the action of $\mathrm{SL}(n, \mathbb{Z})$ preserves ν and there is an infinite subgroup for which the action is free. Thus the orbit equivalence relation induced by $\mathrm{SL}(n, \mathbb{Z})$ is not treeable; since $\cong_n$ (induced by $\mathrm{GL}(n, \mathbb{Q})$) includes $E_{\mathrm{SL}(n,\mathbb{Z})}$ we have that $\cong_n$ is non-treeable. $(\square)$

A similar argument (with thanks to Alekos Kechris and Simon Thomas) shows that $\cong_2$ is not hyperfinite. However, there is no evidence currently to suggest that $\cong_2$ is non-treeable.