Almost disjoint and independent families

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Definition. An infinite family $\mathcal{A} \subseteq [\omega]^\omega$ is *almost disjoint* (*AD*) if any two distinct elements of it have finite intersections.

A family $\mathcal{F} \subseteq [\omega]^\omega$ is *independent* if for any two finite subfamilies $\mathcal{F}_0$, $\mathcal{F}_1 \subseteq \mathcal{F}$ such that $\mathcal{F}_0 \cap \mathcal{F}_1 = \emptyset$ one has $\bigcap \mathcal{F}_0 \setminus \bigcup \mathcal{F}_1 \neq \emptyset$.

Note that $\bigcap \mathcal{F}_0 \setminus \bigcup \mathcal{F}_1 = \bigcap_{F \in \mathcal{F}_0} F \cap \bigcup_{F \in \mathcal{F}_1} \omega \setminus F$.

Proposition 1. (Sierpinski, Hausdorff, Kantorovich-Fichtengol'c)

(1) There is an almost disjoint family of size $\mathfrak{c}$.

(2) There is an independent family of size $\mathfrak{c}$.

Proof. Consider 2^ω and two of its standard bases: $B_0 = \bigcup_{n \in \omega} \{ [s] \mid s \in 2^n \}$, where $[s] = \{ f \in 2^\omega \mid f \restriction n = s \}$, and $B_1 = Clop(2^\omega)$. Let, for $f \in 2^\omega$, $A_f = \{ U \in B_0 \mid f \in U \}$ and $I_f = \{ U \in B_1 \mid f \in U \}$.

Then $\{ A_f \mid f \in 2^\omega \}$ is an AD family (of subsets of B_0) and $\{ I_f \mid f \in 2^\omega \}$ is an independent family (of subsets of B_1).

Definition. Let $\mathcal{A}$ be an AD family and let $\mathcal{C}$, $\mathcal{D}$ be almost disjoint subsets of $\mathcal{A}$. We say that $\mathcal{C}$, $\mathcal{D}$ are *separated* if there is a $Q \in [\omega]^\omega$ such that

$$\forall C \in \mathcal{C}\ C \subseteq^* Q \ \ \& \ \ \forall D \in \mathcal{D}\ D \cap Q =^* \emptyset.$$

(Of course, as usual, $A \subseteq^* B$ if $\overline{\overline{A \setminus B}} < \omega$ and $A =^* B$ if $A \subseteq^* B \ \ \& \ \ B \subseteq^* A$.)

Proposition 2. (Lusin) there is an almost disjoint family, $\mathcal{A}$, of size $\aleph_1$ such that no two disjoint uncountable subsets of $\mathcal{A}$ can be separated.

Proof. Construct $\mathcal{A} = \{ A_\alpha \mid \alpha < \omega_1 \}$ recursively so that for every $\alpha < \omega_1$ and $F \in [\omega]^\omega$ one has

$$\{ \beta < \alpha \mid A_\beta \cap A_\alpha \subseteq F \} \text{ is finite.} (\star)$$

Assume that C separates $\mathcal{C}$ and $\mathcal{D}$ for some uncountable disjoint $\mathcal{C}$, $\mathcal{D} \subseteq \mathcal{A}$. Then there is an uncountable $\mathcal{C}' \subseteq \mathcal{C}$ and a finite set E such that $A_\alpha \setminus C \subseteq E$ for every $A_\alpha \in \mathcal{C}'$. Let $a_\beta \in \mathcal{D}$ be such that $\mathcal{E} = \{ \alpha < \beta \mid A_\alpha \in \mathcal{C}' \}$ is infinite. Then, however, $A_\beta \cap A_\alpha \subseteq E \cup (A_\beta \cap C)$ which contradicts $(\star)$.

Proposition 3. (Matveev). Assuming $\mathrm{MA}_{\sigma\text{-centred}}(\omega_1)$, if $\{ I_\alpha \mid \alpha < \omega_1 \}$ and $\{ J_\alpha \mid \alpha < \omega_1 \}$ are independent families then there is a $\pi \in \mathrm{Sym}(\omega)$ such that $\pi[I_\alpha] =^* I_\alpha$ for all $\alpha < \omega_1$. ($\pi[I] = \pi``I$.)

Definition. An AD family $\mathcal{A}$ is *Luzin* if no two disjoint uncountable subsets of $\mathcal{A}$ can be separated. $\mathcal{A}$ is *anti-Luzin* if for all $\mathcal{C}$, $\mathcal{D}$ which are disjoint uncountable subsets of $\mathcal{A}$ one that that there are $\mathcal{C}' \subseteq \mathcal{C}$, $\mathcal{D}' \subseteq \mathcal{D}$, which can be separated. We will call $\mathcal{A}$ *separated* if any two disjoint subsets of $\mathcal{A}$ can be separated.

Proposition 4. (Roitman-Soukup) Assuming MA(ω_1), every uncountable AD family is either anti-Luzin or contains an uncountable Luzin (sub)family.

Assuming MA, even the structure of Luzin families is somewhat canonical.

Proposition 5. (Abraham-Shelah) Assuming MA, every Luzin AD family can be written as a countable union of subfamilies satisfying property ($\star$).

Proposition 6. (Pospi'sil) $\overline{\overline{\beta\omega}} = 2^{\mathfrak{c}}$.

Proof. It is sufficient to show that there are $2^{\mathfrak{c}}$ incompatible filters on ω. Let $\{\, I_\alpha \mid \alpha < \mathfrak{c} \,\}$ be an independent family of size $\mathfrak{c}$. For $f : \mathfrak{c} \longrightarrow 2$ let $\mathcal{F}_I = \langle \{\, I_\alpha^{f(\alpha)} \mid \alpha < \mathfrak{c} \,\} \rangle$, where $I^0 = I$ and $I^1 = \omega \setminus I$. The family $\{\, I_f \mid f \in 2^{\mathfrak{c}} \,\}$ is the desired family of filters.

Proposition 7. (Hewit-Marczewski-Pordicery) 2^κ is separable for every $\kappa \leq \mathfrak{c}$.

Proof. Let $\{\, I_\alpha \mid \alpha < \kappa \,\}$ be an independent family. Define $f_n(\alpha) = 0$ if and only if $n \in I_\alpha$. If a is a finite partial function from κ to 2 then $f_n \in [s]$ if and only if $n \in \bigcup_{\alpha \in \mathrm{dom}(s)} I_\alpha^{s(\alpha)}$.

Similarly, if $\{\, f_n \mid n \in \omega \,\}$ is dense subset of 2^κ then $\{\, I_\alpha \mid \alpha < \kappa \,\}$, where $I_\alpha = \{\, n \mid f_n(\alpha) = 0 \,\}$, is independent. Properties of independent families translate into properties of the corresponding dense sets and vice versa.

Definition. Let $\mathcal{A}$ be an AD family. Define a topological space $\Psi(\mathcal{A})$ with underlying set $\omega \cup \mathcal{A}$ by declaring points of ω isolated and the base at $A \in calA$ to consist of sets of the form $\{\, A \,\} \cup (A \setminus F)$, where $F \in [\omega]^{<\omega}$.

Proposition 8. $\Psi(\mathcal{A})$ is a locally compact, first countable scattered space.

Proposition 10. (Mrówka) There is an AD family $\mathcal{A}$ such that $\overline{\overline{\mathcal{A}}} = \mathfrak{c}$ and $\overline{\overline{\beta(\Psi(\mathcal{A})) \setminus \Psi(\mathcal{A})}} = 1$.

Proposition 11. (Simon) there are two compact Fréchet spaces whose product is not Fréchet.

Definition. A topological space X is *sequentially separable* if there is a set $D \subseteq X$ such that $\overline{\overline{D}} = \aleph_0$ and for all $x \in X$ there is some $\langle x_n \mid n < \omega \rangle \subseteq D$ such that $x_n \longrightarrow x$. Such a D is sequentially dense.

An independent family $\mathcal{F}$ is *refinable* if for every $f : \mathcal{F} \longrightarrow 2$ there is an $A_f \in [\omega]^\omega$ such that $A_f \subseteq^* I^{f(I)}$ for every $I \in \mathcal{I}$.

A set $X \subseteq \mathbf{R}$ is a *Q-set* if every $Y \subseteq X$ is relative G_δ.

Proposition 11. (Folklore) The following are equivalent:

(1) There is a Q-set of size κ.

(2) There is a separated AD family of size κ.

(3) There is an AD family $\mathcal{A}$ of size κ such that $\Psi(\mathcal{A})$ is normal.

(4) There is a refinable independent family of size κ.

(5) 2^κ is sequentially separable.

Proposition 12. (M.H.-Steprāns) It is relatively consistent with ZFC that 2^{ω_1} is sequentially compact yet not sequentially separable.

Proposition 13. (M.H.-Steprāns) It is relatively consistent with ZFC that 2^{ω_1} is sequentially separable yet there is a countable dense subset of 2^{ω_1} containing no non-trivial convergent sequence.

Definition. A family $\mathcal{S} \subseteq \mathcal{P}(\omega)$ is *splitting* if for every $A \in [\omega]^\omega$ there is an $S \in \mathcal{S}$ such that $\overline{\overline{S \cap A}} = \overline{\overline{A \setminus S}} = \aleph_0$.

A family $\mathcal{S} \subseteq \mathcal{P}(\omega)$ is *independently splitting* if it is both independent and splitting.

$\mathfrak{s} = \min\{\overline{\overline{\mathcal{S}}} \mid \mathcal{S} \text{ is splitting}\}$. $\mathfrak{i}s = \min\{\overline{\overline{\mathcal{S}}} \mid \mathcal{S} \text{ is independently splitting}\}$.

Question. Is $\mathfrak{s} < \mathfrak{i}s$ consistent?

Proposition 14. Every AD family can be extended to a maximal *(MAD)* one. Similarly, every independent family can be extended to a maximal one.

Proposition 15. $\mathcal{A}$ is maximal if and only if $\Psi(\mathcal{A})$ is pseudo-compact. (Of course, not countably compact.)

Definition. A topological space X is *resolvable* if X contains disjoint dense sets. If X is not resolvable we say that X is *irresolvable*. X is *κ-irresolvable* if there is a collection $\{0_\alpha \mid \alpha < \kappa\}$ of pairwise disjoint dense sets. X is *extraresolvable* if there is a collection $\{\mathcal{D}_\alpha \mid \alpha < \cdot(\mathcal{X})^+\}$ of dense sets such that $D_{alpha} \cap D_\beta$ is nowhere dense for $\alpha \neq \beta$, where $\Delta(X) = \min\{\overline{\overline{U}} \mid U \text{ is open in } X\}$.

Proposition 16. Let $\mathcal{I} = \langle I_\alpha \mid \alpha < \kappa \rangle$ be an independent family and let $D_{\mathcal{I}}$ be the corresponding countable dense subset of 2^κ. Then $\mathcal{I}$ is maximal if and only if $D_{\mathcal{I}}$ is irresolvable.

Definition. $\mathfrak{i} = \min\{\overline{\overline{\mathcal{I}}} \mid \mathcal{I} \text{ is a maximal independent family}\}$. $\mathfrak{a} = \min\{\overline{\overline{\mathcal{A}}} \mid \mathcal{A} \text{ is amaximal AD family}\}$.

Proposition 17. (M.H.) Assume $\mathfrak{i} = \mathfrak{c}$. There is a countable ω-resolvable space which is not extraresolvable/

Proposition 18. (M.H.) It is relatively consistent with ZF that every countable space is extraresolvable.

In particular, it is consistent with ZF that there are no maximal independent families on ω.

Question. Is there an irresolvable space in ZF?

Question. (Comfort) Is every pseudo-compact space resolvable?

Note. the answer is yes assuming there are no measurable cardinals (in particular, if $V = L$).

Question. What kinds of MAD families are there?

Definition. Let $\mathbf{P}$ be a forcing notion (partial order) and $\mathcal{A}$ a MAD family. $calA$ is **P**-*indestructible* if $\Vdash_{\mathbf{P}}$ "$\mathcal{A}$ is MAD".

Proposition 19. (M.H.) Let $\mathcal{A}$ be a MAD family.

(1) $\mathcal{A}$ is Cohen-indestructible if and only if $\forall f : \mathbf{Q} \longrightarrow \omega\ \exists A \in \mathcal{A}\ f^{-1}[A]$ is somewhere dense.

(2) $\mathcal{A}$ is Sacks-indestructible if and only if $\forall f : \mathbf{Q} \longrightarrow \omega\ \exists A \in \mathcal{A}\ \overline{f^{-1}[A]}^{\mathbf{R}}$ is uncountable.

Proposition 20. (M.H.) there is a Sacks-indestructible MAD family.

Question. (1) (Steprāns) Is there a Cohen-indestructible MAD family in ZFC? (2) Is there a Sacks-indestructible MAD of size $\mathfrak{c}$ in ZFC?

Definition. A MAD family $\mathcal{A}$ is of *true cardinality* $\mathfrak{c}$ if for every $X \in [\omega]^\omega$ the set $\{A \in \mathcal{A} \mid \overline{\overline{A \cap X}} = \omega\}$ is either finite or of size $\mathfrak{c}$.

Question. (Erdős-Shelah) Is there a MAD family of true cardinality $\mathfrak{c}$ in ZFC?

Proposition 21. (Balcar-Simon) the answer to the above is positive assuming one of the following:

$$\mathfrak{a} = \mathfrak{c} \ , \ \mathfrak{s} = \omega_1 \ , \ \mathfrak{b} = \mathfrak{d} \ , \ \mathfrak{d} \leq \mathfrak{a},$$

where $\mathfrak{b} = \min\{\overline{\overline{\mathcal{F}}} \mid \mathcal{F} \subseteq \omega^\omega \ \ \& \ \ \forall f \in \omega^\omega \ \exists g \in \mathcal{F}\ \} \not\leq^* \{\}$ and $\mathfrak{b} = \min\{\overline{\overline{\mathcal{F}}} \mid \mathcal{F} \subseteq \omega^\omega \ \ \& \ \ \forall f \in \omega^\omega \ \exists g \in \mathcal{F} \ \{\ \leq^* \}$.

Question. (Roitman) Does $\mathfrak{d} = \omega_1$ imply $\mathfrak{a} = \omega_1$?

Definition. We say that $\diamondsuit_d$ holds if there is a sequence $\langle d_\alpha \mid \alpha < \omega_1 \rangle$ such that $d_\alpha : \alpha \longrightarrow \omega$ and for all $f : \omega_1 \longrightarrow \omega$ there is some $\alpha \geq \omega$ such that $f \restriction \alpha \leq^* f_\alpha$.

Proposition 22. (M.H.) $\diamondsuit_d$ implies $\mathfrak{a} = \omega_1$.