# A Nonstandard Approach to Baire Category

Set theory and game theory
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**Introduction.**

This talk presents my attempts to understand the ideas of Baire category from a 'logical point of view', using nonstandard analysis as a tool.

As quoted in *Models of Peano arithmetic*:

'Consider it logically,' Valdez said. 'What is my problem? To find Cathy. What do I know about Cathy? Nothing.'

'That doesn't sound so good,' Marvin said.

'But it is only half of the problem. Granted that I know nothing about Cathy, what do I know about Finding?'

'What?' Marvin asked.

'It happens that I know everything about Finding,' Valdez said triumphantly, gesturing with his graceful terracotta hands. 'For it happens that I am an expert in the Theory of Searches!'

Robert Sheckley
*Mindswap*

The starting point is the well-known observation that measure and category are analogous (or 'dual').

Standard result in the literature:

**Theorem** [Sierpiński–Erdős, CH.] There is a bijection $f\colon \mathbb{R} \to \mathbb{R}$ such that $f^{-1} = f$ and for any set $A \subseteq \mathbb{R}$,

$$A \text{ has measure } 0 \Leftrightarrow f''A \text{ is meagre.}$$

However this does not explain the situation completely.

In particular there is no analogue of the ‘measure’ function itself!

Since Lebesgue measure has a particularly nice presentation using nonstandard analysis (Löb), it’s natural to look for analogues of measure for Baire category using NSA, and based on combinatorial principles in nonstandard discrete spaces.

**‘Measure’.**

If measure and category are dual, a ‘measure’ for Baire categorical notions would be *dual* to Lebesgue measure.

I shall use the term ‘comeasure’ for the dual of ‘measure’. But I will only use this term informally.

*Future project (?)* An axiomatic approach to measure and comeasure.

**Expected differences between measure and comeasure.**

1. Measure takes values in a set with additive algebraic structure and order relation $\leqslant$, whereas comeasure may be expected to have a multiplicative algebraic structure.

2. Measure is additive for disjoint sets, and more generally we have

$$m(A \cup B) \leqslant m(A) + m(B),$$

on the other hand, comeasures would have

$$m(A \cap B) \geqslant m(A) \cdot m(B).$$

**Large and small sets.**

Let $X$ be a set.

A *notion of large subset* of $X$ is a collection $L \subseteq \mathscr{P}X$ such that:

1. $A \supseteq B \in L$ implies $A \in L$; and
2. $A \in L$ implies $A \neq \varnothing$.

A *notion of small subset* of $X$ is a collection $S \subseteq \mathscr{P}X$ such that:

1. $A \subseteq B \in S$ implies $A \in S$; and
2. $A \in S$ implies $A \neq X$.

$$L^c =_{\mathsf{def}} \{A \subseteq X : A^c \in L\}$$

**Dual.**

$L^\dagger =_{\text{def}} \{A \subseteq X : A^c \notin L\}$.

Equivalently,

$L^\dagger =_{\text{def}} \{A \subseteq X : \forall B{\in}L \;\; A \cap B \neq \varnothing\}$.

E.g., if $X = [0,1]$, take for $L$ the sets of Lebesgue measure 1. Then $L^\dagger$ is the sets of *nonzero outer measure*.

**Sample space.**

Let $X = [0,1]$, and fix a sufficiently saturated nonstandard universe ${}^*V$.

Fix some $N \in {}^*\mathbb{N} \setminus \mathbb{N}$, and let

$$\Omega =_{\text{def}} \{n : n \in {}^*\mathbb{N}, 0 \leqslant n < N\},$$
$$\Omega/N =_{\text{def}} \{n/N : n \in \Omega\}.$$

The *standard part map* $\Omega/N \to X$ is denoted by st. (For this talk, this always has domain $\Omega/N$, so for $A \subseteq X$, $\text{st}^{-1}A$ is unambiguous.)

A set $A \subseteq \Omega/N$ is *internal* if it is the extension of a set in ${}^*V$.

It is *convex* if $\forall u, v, w \in \Omega/N\ (u{<}v{<}w\ \&\ u, w{\in}A \Rightarrow v{\in}A)$.

**Lebesgue measure.**

The *measure*, $\mu(B)$, of an internal $B \subseteq \Omega/N$ is $^*\text{card}(B)$. The *normalized* measure, $\widehat{\mu}(B)$, is

$$\frac{^*\text{card}(B)}{^*\text{card}(\Omega/N)} = \frac{^*\text{card}(B)}{N}.$$

Also define

$$^\circ\mu(B) =_{\text{def}} \text{st}\,\widehat{\mu}(B).$$

A set $A \subseteq X$ has *outer Lebesgue measure*

$\overline{\lambda}(A) = \inf\{^\circ\mu(B) : B \subseteq \Omega \text{ internal, } B \supseteq \text{st}^{-1}A\}$

and *inner Lebesgue measure* given by

$\underline{\lambda}(A) = \sup\{^\circ\mu(B) : B \subseteq \Omega \text{ internal, } B \subseteq \text{st}^{-1}A\}$.

If these agree then $A$ is *Lebesgue measurable* with Lebesgue measure $\lambda(A) = \underline{\lambda}(A) = \overline{\lambda}(A)$.

**Baire category.**

Fix $X = [0, 1]$, with the usual topology.

A *corare* set is one whose interior is dense in $X$. A *rare* set is the complement in $X$ of a corare set.

A *comeagre* set is the intesection of countably many corare sets. A *meagre* set is the complement in $X$ of a comeagre set.

Alternative terminology:

1. comeagre = residual
2. meagre = 1st category
3. not meagre = 2nd category

A set has the *property of Baire* if it is the symmetric difference of an open set with a meagre set.

**A comeasure for corare sets.**

For $x \in \Omega$, an *$x$-interval* of $\Omega/N$ is an internal convex $I \subseteq \Omega/N$ with $^*\text{card}(I) = x$.

A *$y$-subinterval* of such $I$ is $B \subseteq I$ which is also a $y$-interval.

*Definition.* If $B \subseteq \Omega/N$ is internal, $\delta(B)$ is the internal function $\delta(B)\colon \Omega \to \Omega$ defined by

$$x \mapsto \max\left\{ y : \begin{array}{l} \text{every } x\text{-interval of } \Omega/N \\ \text{contains a } y\text{-subinterval } I \subseteq B \end{array} \right\}.$$

By convention, $\delta(B)(0) = 0$.

The normalized version $\hat{\delta}(B)\colon \Omega/N \to \Omega/N$ is

$$\hat{\delta}(B)(x) = (\delta(NB)(Nx))/N$$

where $NB = \{Nb : b \in B\}$.

**Lemma.** For $x, y \in \Omega$ and internal $B, C \subseteq \Omega$:

i) $x \leqslant y \Rightarrow \delta(B)(x) \leqslant \delta(B)(y)$;

ii) $0 \leqslant \delta(B)(x) \leqslant x$;

iii) if $B \subseteq C$ then $\delta(B)(x) \leqslant \delta(C)(x)$;

iv) $\delta(B \cap C)(x) \geqslant \delta(B) \circ \delta(C)\ (x)$;

v) $\delta(B)(x) \leqslant \delta(B)(x+1) \leqslant \delta(B)(x) + 1$;

vi) $\delta(B)(x) \leqslant \delta(B)(x+y) \leqslant \delta(B)(x) + y$.

For internal $B \subseteq \Omega/N$, ${}^{\circ}\!\delta(B)\colon [0,1] \to [0,1]$ is defined by ${}^{\circ}\!\delta(B)(\operatorname{st} x) = \operatorname{st} \widehat{\delta}(B)(x)$.

$\mathfrak{F} =_{\text{def}}$ set of continuous functions $f\colon [0,1] \to [0,1]$ such that

$$0 \leqslant f(x_1) - f(x_0) \leqslant x_1 - x_0$$

for all $x_0 \leqslant x_1$ in $[0,1]$. (Such $f$ are necessarily nondecreasing with $f(0) = 0$.)

For $f, g \in \mathfrak{F}$,
$f \leqslant g \Leftrightarrow f(x) \leqslant g(x)$ for all $x \in [0,1]$
$f \circ g$ is the composite $x \mapsto f(g(x))$.

**Lemma.** For internal $B, C \subseteq \Omega$:

i) ${}^{\circ}\!\delta(B) \in \mathfrak{F}$;

ii) if $A \subseteq B$ then ${}^{\circ}\!\delta(A) \leqslant {}^{\circ}\!\delta(B)$;

iii) ${}^{\circ}\!\delta(A \cap B) \geqslant {}^{\circ}\!\delta(A) \circ {}^{\circ}\!\delta(B)$;

**$\rho$-comeasure.**

Define the *outer comeasure* of $A \subseteq [0,1]$ by

$$\overline{\rho}(A) = \bigwedge\{{}^\circ\delta(B) : B \subseteq \Omega \text{ internal, } B \supseteq \mathrm{st}^{-1}A\}$$

and the *inner comeasure* by

$$\underline{\rho}(A) = \bigvee\{{}^\circ\delta(B) : B \subseteq \Omega \text{ internal, } B \subseteq \mathrm{st}^{-1}A\}.$$

These are well-defined, e.g., because

$$\overline{\rho}(A)\,(x) = \inf\left\{{}^\circ\delta(B)(x) \;:\; \begin{array}{l} B \subseteq \Omega \text{ internal} \\ B \supseteq \mathrm{st}^{-1}A \end{array}\right\}.$$

$\rho(A)$ is the common value of these two when they agree.

**Results on the $\rho$-comeasure.**

1. Every $A \subseteq X$ is $\rho$-comeasurable.

2. $A$ is dense in $X$ iff ${}^\circ\delta(B)(x) > 0$ for all internal $B \supseteq \mathrm{st}^{-1}A$ and all $x > 0$.

3. $A$ is corare in $X$ iff there is $f \in \mathfrak{F}$ such that $f(x) > 0$ for all $x > 0$, and ${}^\circ\delta(B) \geqslant f$ for all internal $B \supseteq \mathrm{st}^{-1}A$.

[Compare with:
$A$ has nonzero outer Lebesgue measure if there is $\epsilon > 0$ such that ${}^\circ\mu(B) \geqslant \epsilon$ for all internal $B \supseteq \mathrm{st}^{-1}A$.]

Let $\mathfrak{G}$ be the set of *germs* of $f \in \mathfrak{F}$ at 0, that is, $\mathfrak{G} = \mathfrak{F}/\sim$ where

$$f \sim g \Leftrightarrow \exists y > 0\ \forall x < y\ f(x) = g(x).$$

4. $A$ is corare in $X$ iff there is $g \in \mathfrak{G}$ such that $g > 0$ and ${}^\circ\delta(B)/\sim\ \geqslant g$ for all internal $B \supseteq \mathrm{st}^{-1}A$.

**Comeagre sets.**

There is a problem applying this to comeagre sets!

Our $\mathfrak{G}$ has the *'no-zero divisors property'*,

$$f/\!\sim \circ\, g/\!\sim\, = 0 \Rightarrow f/\!\sim\, = 0 \text{ or } g/\!\sim\, = 0,$$

but not the *'no $\sigma$-zero divisors property'*,

$$\bigwedge_{n\in\omega} (f_n/\!\sim \circ \cdots \circ f_1/\!\sim \circ\, f_0/\!\sim) = 0$$
$$\Rightarrow f_i/\!\sim\, = 0 \text{ for some } i \in \omega.$$

One solution might be to 'artificially' complete $\mathfrak{G}$.

**Definition.** $\mathfrak{H}$ is the set $\omega$-sequences $(g_i)_{i\in\omega}$ from $\mathfrak{G}$ which are *monotonic nonincreasing* $(i \leqslant j \Rightarrow g_i \geqslant g_j)$ modulo the equivalence $\equiv$ given by

$$(g_i)_{i\in\omega} \preceq (h_i)_{i\in\omega} \Leftrightarrow \forall i\ \exists j > i\ g_j \leqslant h_i$$

and

$$(g_i) \equiv (h_i) \Leftrightarrow (g_i) \preceq (h_i)\ \&\ (h_i) \preceq (g_i)$$

**Definition.**

$$\underline{\alpha}(A) = \bigvee \left\{ (g_i)/\!\equiv\ :\ \begin{array}{c} A = \bigcap_{i\in\omega} A_i \\ A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \\ g_i = \rho(A_i)/\!\sim \end{array} \right\}$$

**Theorem.** $A \subseteq [0,1]$ is comeagre iff there is $h > 0$ in $\mathfrak{H}$ such that

$\underline{\alpha}(B) \geqslant h$ for all $B \supseteq A$ for which $\underline{\alpha}(B)$ is defined.

*Question:*
For which $B \subseteq [0,1]$ *is* $\underline{\alpha}(B)$ defined?

**A nonstandard version of the Banach–Mazur game.**

Let $\mathscr{I}$ be the set of all internal intervals of $\Omega/N$, and let $\mathscr{I}^+$ be the set of all $I \in \mathscr{I}$ with ${}^\circ\mu I > 0$.

These correspond to 'standard' intervals via the standard-part map

$${}^\circ I = \{\mathsf{st}(x) : x \in I\}$$

and

$${}^\circ\mathscr{I} = \{{}^\circ I : I \in \mathscr{I}\}$$

**Theorem.** $A \subseteq X$ is comeagre iff there is an internal $f\colon \mathscr{I} \to \mathscr{I}$ such that:

1. $f(I) \subseteq I$ for all $I$;
2. if $I \in \mathscr{I}^+$ then $f(I) \in \mathscr{I}^+$;
3. whenever $B \supseteq \mathrm{st}^{-1}A$ is internal and

$$I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$$

is an internal sequence of intervals in $\mathscr{I}^+$ with $I_{i+1} \subseteq f(I_i)$ for all $i$ then there is $n \in \mathbb{N}$ such that $I_n \subseteq B$.

The function $f$ can always be chosen so that

$$f(I) \subseteq J \subseteq I \Rightarrow f(I) = f(J)$$

and for any maximal chain of open intervals $\mathscr{C} \subseteq {}^\circ\mathscr{I}$, the map $\mathscr{C} \to [0,1]$ given by

$${}^\circ I \mapsto \lambda({}^\circ f(I))$$

is well-defined and continuous.

**Internal functions on intervals.**

Suppose $f\colon \mathscr{I} \to \mathscr{I}$ is internal such that

1. $f(I) \subseteq I$ for all $I$;

2. for an end-point $x$ of $I$ and $J = I \setminus \{x\}$, *either* $f(I) = f(J)$ *or* $x \in f(I)$ and

$$^*\mathrm{card} f(J) + 1 \geqslant {}^*\mathrm{card} f(I) \geqslant \mathrm{card}\, f(J);$$

Then

$$\widehat{\delta}(f)\colon \Omega/N \to \Omega/N$$

may be defined by

$$\widehat{\delta}(f)(x) = \min\{\widehat{\mu}(fI) : I \in \mathscr{I},\ \widehat{\mu}(I) \geqslant x\}.$$

The function ${}^\circ\delta(f)\colon [0,1] \to [0,1]$ defined by

$${}^\circ\delta(f)(\mathrm{st}\, x) = \mathrm{st}\, \widehat{\delta}(f)(x)$$

is well-defined, continuous, and in $\mathfrak{F}$. It seems that ${}^\circ\delta(f)/\sim$ gives a good 'measure' of $f$.

**Strategies for $A \subseteq X$.**

Say $f\colon \mathscr{I} \to \mathscr{I}$ is a *strategy for* $A \subseteq X$ if:

1. $f(I) \subseteq I$ for all $I$;

2. $I \subseteq J$ implies $f(I) \subseteq f(J)$ for all $I, J$;

3. whenever $B \supseteq \mathrm{st}^{-1}A$ is internal and

$$I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$$

is an internal sequence of intervals in $\mathscr{I}$ with $I_{i+1} \subseteq f(I_i)$ for all $i$ then there is $n \in \mathbb{N}$ s.t.

*either* $I_{n+1} \notin \mathscr{I}^+$ and $f(I_i) \in \mathscr{I}^+$
*or* $I_n \subseteq B$;

4. $f$ has the 'continuity' property on the previous slide.

Define,

$\underline{\beta}(A) =_{\mathrm{def}} \bigvee \{ {}^\circ\delta(f)/\!\sim\, : f \text{ is a strategy for } A\}$

**Work remaining.**

We just defined an 'inner' comeasure characterizing the comeagre sets. An 'outer comeasure' was expected, and is needed to develop these ideas further.

Something like

$$\overline{\beta}(A) =_{\mathsf{def}} \bigwedge\{{}^\circ\delta(f)/\sim : f \text{ covers } A\}$$

(But what does 'covers' mean?)

Or,

$$\overline{\beta}(A) =_{\mathsf{def}} \bigwedge\left\{{}^\circ\delta(f)/\sim : \begin{array}{l} f \text{ is a best strategy} \\ \text{for some } B \supseteq \mathrm{st}^{-1}A \end{array}\right\}$$

(But what are these nonstandard finite games, and what does 'best' mean?)