A New Fine-Structure Theory for Constructible Inner Models

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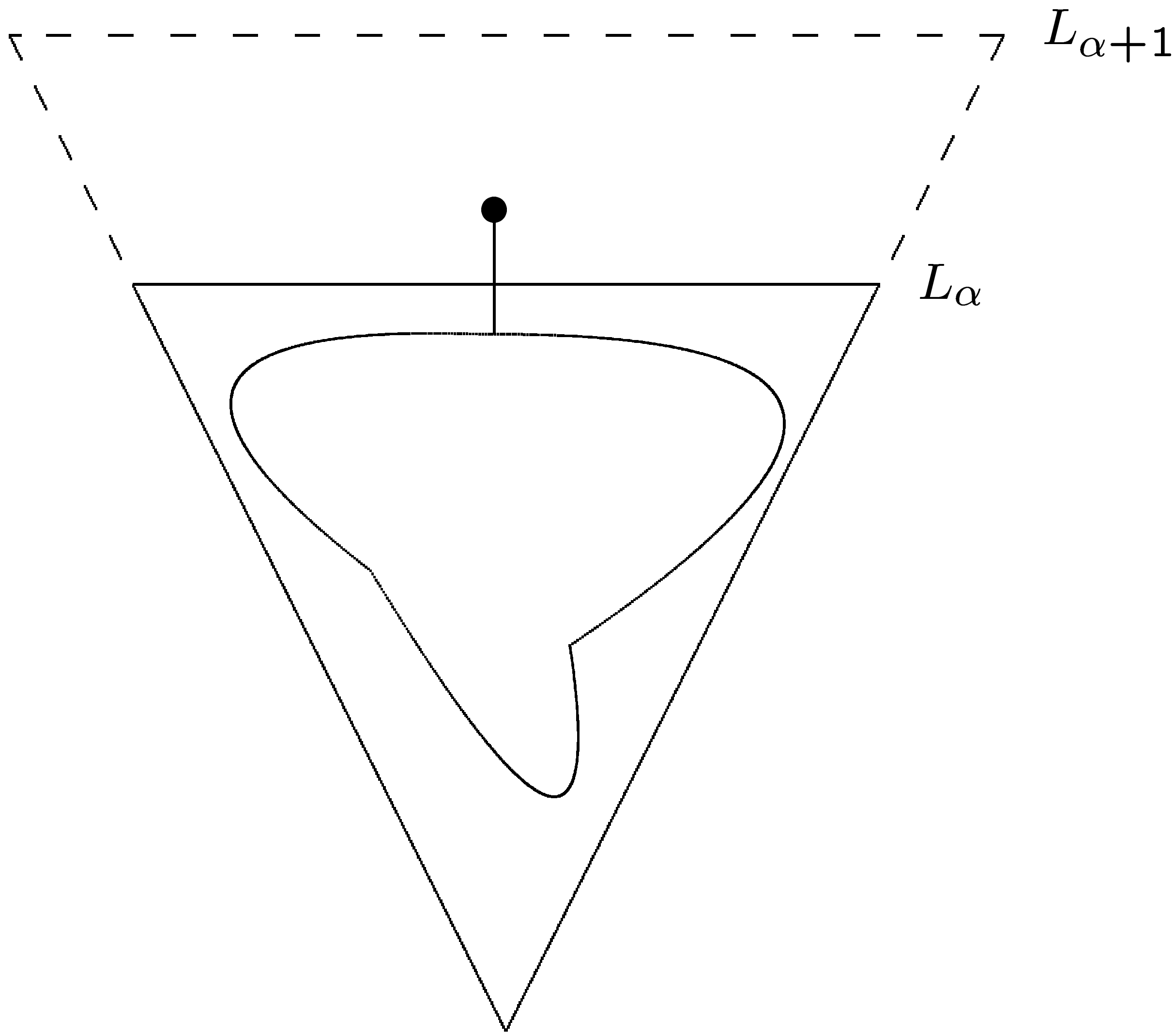
I. A Reminder of Gödel's Model L

II. The New Hierachy

III. A "Proof" of Jensen's Covering Theorem

I. A Reminder of Gödel's Model L

$$L = \bigcup_{\alpha \in \mathrm{On}} L_\alpha,\ L_0 = \emptyset$$



The Interpretation Function

- $I(L_\alpha, \varphi, \vec{y}) = \{x \in L_\alpha | (L_\alpha, \in) \models \varphi[x, \vec{y}]\}$

- $L_{\alpha+1} = \{I(L_\alpha, \varphi, \vec{y}) | \varphi \text{ formula}, \vec{y} \in L_\alpha\}$

- λ limit $\rightarrow L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$

The Gödel Hierachy satisfies

- $\alpha < \beta \rightarrow L_\alpha \subseteq L_\beta$
- $\alpha < \beta \rightarrow L_\alpha \in L_\beta$
- L_α transitive
- $L \models$ Zermelo-Fraenkel set theory

Wellordering L

$a <_L b :\equiv$ "least" triple $(L_\alpha, \varphi, \vec{y})$ such that $a = I(L_\alpha, \varphi, \vec{y})$ is lexicographically smaller than "least" triple $(L_\beta, \varphi, \vec{z})$ such that $b = I(L_\beta, \psi, \vec{z})$.

- $L \models$ Axiom of Choice

Skolem function

$$S(L_\alpha, \varphi, \vec{y}) \simeq \text{the } <_L - \text{least } x \text{ such that } L_\alpha \models \varphi[x, \vec{y}]$$

- $L \models$ Generalized Continuum Hypothesis
- $L \models \diamondsuit$

Jensen's Fine-Structure Theory

- I, $<_L$, S are definable in low complexity:

  $S(L_\alpha, \varphi, -)$ for Σ_1-fomulas φ is $\Sigma_1(L_\alpha)$

- $\Sigma_n(L_\alpha)$ is analysed as n iterations of Σ_1-definability.

Jensen

- $L \models \square$
- $L \models$ Morasses exist
- Covering Theorem for L

II. The New Hierachy

The Approach

- Make I, $<$, S, $\in$ symbols of a new language $\mathcal{S}$.

- Define hierachy
$\mathcal{F}_\alpha = (F_\alpha, I \restriction F_\alpha, <\restriction F_\alpha, S \restriction F_\alpha, \in)$.

- $F_{\alpha+1}$ is obtained by quantifier-free $\mathcal{S}$-definability.

The Hierachy

$\mathcal{S}_0 =$ quantifier-free $\mathcal{S}$ formulae

$I(F_\alpha, \varphi, \vec{y}) = \{x \in F_\alpha | \mathcal{F}_\alpha \models \varphi[x, \vec{y}]\}$

$F_{\alpha+1} = \{I(F_\alpha, \varphi, \vec{y}) | \varphi \in \mathcal{S}_0, \vec{y} \in F_\alpha\}$

(Ternary) Wellordering and Skolem Function

$a <_{F_\alpha} b :\equiv$ lex. least $(F_\beta, \varphi, \vec{y})$ with $a = I(F_\beta, \varphi, \vec{y})$ is lex. smaller than lex. least $(F_\gamma, \psi, \vec{z})$ with $b = I(F_\gamma, \psi, \vec{z})$.

$S(F_\alpha, \varphi, \vec{y}) \simeq$ the $<_{F_\alpha}$-least x such that $\mathcal{F}_\alpha \models \varphi[x, \vec{y}]$

The Fine Hierachy

- $\mathcal{F}_0 = (\emptyset, \dots)$

- $\mathcal{F}_{\alpha+1} = (F_{\alpha+1}, I \restriction F_{\alpha+1}, S \restriction F_{\alpha+1}, < \restriction F_{\alpha+1}, \in)$

- limit $\lambda \to \mathcal{F}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{F}_\alpha$

- $L = \bigcup_{\alpha \in \mathsf{On}} F_\alpha$

Properties of the Fine Hierachy $(\mathcal{F}_\alpha)_{\alpha \in \mathbf{On}}$

- hierachical properties as for L
- $F_{\omega\alpha} = J_\alpha$ *(Jensen hierachy)*
- Condensation:
  If $X \subseteq \mathcal{F}_\alpha$ is a substructure of $\mathcal{F}_\alpha$
  then $X \cong \mathcal{F}_{\bar{\alpha}}$ for some $\bar{\alpha} \leq \alpha$

Direct Limits

If the directed System $(\mathcal{F}_{\alpha_i}), (\pi_{ij})$ of injective homomorphisms $\pi_{ij} : \mathcal{F}_{\alpha_i} \to \mathcal{F}_{\alpha_j}$ has a well-founded limit, the limit is isomorphic to some $\mathcal{F}_{\alpha}$.

III. A "Proof" of Jensen's Covering Theorem

The Covering Theorem

Assume $O^{\#}$ does not exist. Then L covers V, i.e.,

$$\forall X \subseteq \mathsf{On}\, \exists Z \in L\, (X \subseteq Z \wedge \bar{\bar{Z}} \leq \bar{\bar{X}} + \aleph_1)$$

Corollary

If $O^{\#}$ does not exist, then

- every singular cardinal κ in V is singular in L

- moreover: $(\kappa^+)^V = (\kappa^+)^L$

"Proof" (by Contradiction)

Let X be minimal counterexample to covering:
$\tau = \mathrm{sup}(X)$ minimal, card(X) minimal.

$O^{\#} \simeq$ there exists a non-trivial elementary $L \to L$.

Structure of the Proof

$$
\begin{tikzcd}
L \arrow[r, dashed, "\tilde{\sigma}"] & L \\
\mathcal{F}_{\bar{\tau}} \arrow[u, "\subseteq"'] \arrow[r, "\sigma"] \arrow[dr, "\cong"] & \mathcal{F}_{\tau} \arrow[u, "\subseteq"'] \\
 & Y \arrow[u, "\prec"'] \\
 & X \arrow[u, "\subseteq"']
\end{tikzcd}
$$

The Extension of $\mathcal{F}_\delta$ by σ

$$\begin{tikzcd}
F_\delta \arrow[r, dashed, "\tilde{\sigma}"] & F_{\tilde{\delta}} \\
\mathcal{F}_{\bar{\tau}} \arrow[u, "\subseteq"'] \arrow[r, "\sigma"] & \mathcal{F}_\tau \arrow[u, "\subseteq"']
\end{tikzcd}$$

ASSUMPTION

$\forall \mu < \bar{\tau}\ \forall$ finite $p \subseteq \mathcal{F}_\delta$:
the substructur of $\mathcal{F}_\delta$ generated by $\mu \cup p$ is isomorphic to some $\mathcal{F}_{\alpha(\mu,p)}$, $\alpha(\mu,p) \leq \bar{\tau}$

$$\pi_{\mu p} : \mathcal{F}_{\alpha(\mu,p)} \cong \langle \mu \cup p \rangle \subseteq \mathcal{F}_\delta$$

Lifting a Directed System

$$\begin{tikzcd}[column sep=huge]
\mathcal{F}_\delta \arrow[rr, dashed, "\tilde{\sigma} \supseteq \sigma"] & & \mathcal{F}_{\tilde{\delta}} \\
\mathcal{F}_{\alpha(\nu,q)} \arrow[u, "\pi_{\nu p}"'] & & \mathcal{F}_{\alpha^*(\nu,q)} \arrow[u, "\pi^*_{\nu q}"'] \\
\mathcal{F}_{\alpha(\mu,p)} \arrow[u, "\pi_{\mu p,\nu q}"'] \arrow[rr, phantom, "\overset{\sigma}{\longmapsto}"] & & \mathcal{F}_{\alpha^*(\mu,p)} \arrow[u, "\pi^*_{\mu p,\nu q}"']
\end{tikzcd}$$

$\in \mathcal{F}_{\bar{\tau}}$ $\qquad\qquad$ $\in \mathcal{F}_{\tau}$

If $\delta = \infty$ satisfies the ASSUMPTION then $0^{\#}$ exists, CONTRADICTION.

Otherwise let δ be maximal satisfying the ASSUMPTION.

- $\mathcal{F}_{\delta+1}$ is generated from some $\nu \cup q$

$$F_{\delta+1} = \langle \nu \cup q \rangle$$

Pushing up a $\exists_1$-Elementary Map

$$\begin{tikzcd}
\tilde{\sigma} : \mathcal{F}_\delta \arrow[r] \arrow[d, "\subseteq"] & \mathcal{F}_{\tilde{\delta}} \arrow[d, "\subseteq"] & \text{is } \exists_1\text{-elementary} \\
\tilde{\sigma}^+ : \mathcal{F}_{\delta+1} \arrow[r] & \mathcal{F}_{\tilde{\delta}+1} & \text{is a homorphism}
\end{tikzcd}$$

$$\begin{aligned} X &\subseteq \text{range}(\sigma) \subseteq \text{range}(\tilde{\sigma}^{+}) \\ &= \tilde{\sigma}^{+}[\langle \nu \cup q \rangle] \\ &= \langle \tilde{\sigma}^{+}[\nu] \cup \tilde{\sigma}^{+}[q] \rangle \\ &\subseteq \langle \tilde{\sigma}^{+}(\nu) \cup \tilde{\sigma}^{+}[q] \rangle =: W \end{aligned}$$

where $W \in L$, $\text{card}^{L}(W) \leq \tilde{\sigma}^{+}(\nu) < \tau$

Get covering set for X, CONTRADICTION.

Further Applications

- $\Box$, Morasses (Hyperfine structure theory, Friedman, K.)

- *Coremodels:* $\tilde{\sigma} : \mathcal{F}_\delta \to \mathcal{F}_{\tilde{\delta}}$ corresponds to a fine-structural ultrapower at $\mathcal{F}_\delta$ by the "extender" σ.