Dimension Theory inside a Homogeneous Model

Olivier Lessmann

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What is Dimension Theory?

Contexts with a good dimension theory:

- **Linear Algebra**;
  Two vector spaces over the same field are isomorphic iff they have the same dimension.

- **Field Theory**;
  Two algebraically closed fields of the same characteristics are isomorphic iff they have the same transcendence degree over the prime field.

- **First order categoricity**;
  Baldwin-Lachlan Theorem for countable, first-order theories.

Definition 1 *A* **pregeometry** *is a pair* (W, cl)*, where*

$$cl : \mathcal{P}(W) \to \mathcal{P}(W)$$

satisfies

1. $x \in cl(X)$*, for each* $x \in X$*.*

2. *If* $x \in cl(X)$ *and* $X \subseteq cl(Y)$*, then* $x \in cl(Y)$*.*

3. *If* $x \in cl(X)$*, there exists a finite* $Y \subseteq X$ *such that* $x \in cl(Y)$*.*

4. *(Exchange)*
   If $x \in cl(Xy) \setminus cl(X)$*, then* $y \in cl(Xx)$*.*

Inside a pregeometry, we can define independence, generating sets, bases, and do 'dimension theory'.

Homogeneous mathematical structures

First approximation:

A structure M is **homogeneous** if any partial isomorphism $f : M \to M$ with $|f| < \|M\|$ extends to an automorphism of M.

Definition 2 *M is* **homogeneous** *if any partial elementary embedding $f : M \to M$ with $|f| < \|M\|$ extends to an automorphism of M.*

We can replace **elementary embeddings** by embeddings preserving a prescribed set of formulas (e.g. quantifier free, existential, formulas in n variables, ...).

Homogeneous vs. Saturated Model Theory

First order model theory is concerned with the class of elementary submodels of a big **saturated** model $\mathfrak{C}$. This class corresponds to the class of models of $Th(\mathfrak{C})$. $\mathfrak{C}$ is a universal domain for the class of (small) models of $Th(\mathfrak{C})$, as they all embed elementarily inside $\mathfrak{C}$.

Homogeneous model theory is concerned with the class of elementary submodels of a big **homogeneous** model $\mathfrak{C}$. This class corresponds to the class of models of $Th(\mathfrak{C})$ omitting all the types omitted by $\mathfrak{C}$. $\mathfrak{C}$ is a universal domain for the class of (small) models of $Th(\mathfrak{C})$ omitting (over the empty set) the types omitted by $\mathfrak{C}$, as they all elementarily embed inside $\mathfrak{C}$.

Saturated models are homogeneous, so homogeneous model theory is more general than first order model theory. We lose the compactness theorem at this level of generality.

Advantage of Homogeneous Model Theory

Very general, much of current model theory falls within homogeneous model theory.

Examples

- First order case (saturated models)
- Robinson theories (universal domains)
- e.c. models of a universal (inductive) first order theory (e-universal domains)
- Banach space model theory ('saturated models')
- Finite variable logic with amalgamation over sets (n-variable saturated)

- Generic structures
  - Hrushovski constructions, Zilber
  - Generic automorphisms
  - (M, N) with $M \prec M$
- Class of models with amalgamation over sets (e.g. the class of models of a f.o. theory omitting a prescribed set of types, infinitary logics, etc.)
- Concrete examples like Hilbert spaces, normed fields, etc.

Understanding types

Let $A \subseteq M$. A **type over** A **in** M is a set of formulas with parameters over A, each finite subset of which is realized in M.

Inside a saturated model M, all types over subsets of M of size less than $\|M\|$ are realized in M. This is **compactness**.

This fails inside a homogeneous model. Which types are realized inside a homogeneous $\mathfrak{C}$?

A **complete type over** A **in** M is a maximal type over A in M, i.e. it contains each formula over A or its negation.

Inside a homogeneous model M, all **complete** types p over $A \subseteq M$ of size less than $\|M\|$ are realized in M, **provided** $p|B$ is realized in M, for each finite subset $B \subseteq A$. We refer to this as **weak compactness**.

We write $tp(\bar{a}/B, M)$ for the complete type consisting of the formulas over B realized by $\bar{a}$ in M.

Consequences of weak compactness for homogeneous model theory

Let $\mathfrak{C}$ be a big homogeneous model.

Fact 1 *If* $(p_i : i < \alpha < \|\mathfrak{C}\|)$ *is an increasing sequence of complete types* p_i*, each of which is realized in* $\mathfrak{C}$*, then the union is a complete type realized in* $\mathfrak{C}$*.*

Fact 2 *Let* I *and* A *be in* $\mathfrak{C}$*. If* I *is large enough compared to* A*, then there exists an* A*-indiscernible sequence* J *inside* $\mathfrak{C}$ **similar** *to* I*.*

Fact 3 *Let* I *be in* $\mathfrak{C}$*. If* I *is* A*-indiscernible, then* I *can be* **extended** *to the size of* $\mathfrak{C}$*.*

Categoricity

Definition 3 *A class of models $\mathcal{K}$ is* **λ-categorical** *if all models of $\mathcal{K}$ of cardinality λ are isomorphic.*

Why study categoricity?

Zil'ber: Categoricity is the logician's way of seeing canonicity. Meaningful and important mathematical structures are categorical, or live inside categorical structures. Conversely, mathematical structures which are categorical have a very exact, yet concise, description, and must be of special interest in mathematics.

Shelah: Main test question to develop powerful model-theoretic techniques. Stability theory (= very general dimension theory) for first order theory was a continuation of Morley's uncountable categoricity theorem. Categoricity for nonelementary classes is the most important problem in model theory.

Theorem 1 (Baldwin-Lachlan) *Assume that the class of models of a countable first order theory T is λ-categorical for some uncountable λ. Then*

- *Each model is prime and minimal over a strongly minimal set (and its parameters).*

- *The dimension of the pregeometry induced by the strongly minimal set determines the isomorphism-type of the model and for uncountable models, the dimension of the pregeometry equals the size of the model.*

Hence the class of models of T is μ-categorical in every uncountable μ.

Let L be countable and let $\mathfrak{C}$ be a big homogeneous L-structure. Let $\mathcal{K}$ be the class of elementary submodels of $\mathfrak{C}$.

Theorem 2 (Lessmann) *Assume $\mathcal{K}$ is λ-categorical for some uncountable λ. Then*

- *Each uncountable model in $\mathcal{K}$ is prime and minimal over a quasi-minimal set (and its parameters).*

- *The dimension of the pregeometry induced on the quasi-minimal set determines the isomorphism-type of the model and, for uncountable models, the dimension of the pregeometry equals the size of the model.*

Hence $\mathcal{K}$ is μ-categorical for all uncountable μ.

Definition 4 *We say that a 1-type $p(x)$ over finitely many parameters is* **quasi-minimal** *if $p(x)$ is realized uncountably many times in $\mathfrak{C}$, but for each formula $\phi(x, b)$, exactly one of*

$$p(x) \cup \{\phi(x, b)\} \quad \textit{or} \quad p(x) \cup \{\neg\phi(x, b)\}$$

is realized at most countably many times.

A **quasi-minimal set** in a model M is the set of realizations in M of a quasi-minimal type (over M).

Quasi-minimal types generalize first order strongly minimal types. An equivalent concept was introduced by Shelah in the 1970s. The term was coined in the 1990s by Zilber, who used quasi-minimality in his work around the complex numbers with exponentiation.

First step

Proposition 1 *Suppose $\mathcal{K}$ is λ-categorical for some uncountable λ. Then*

- *Quasi-minimal sets exist in each uncountable model $M \in \mathcal{K}$.*

- *If W is a quasi-minimal set in some uncountable model $M \in \mathcal{K}$, then (W, cl) is a pregeometry, where*

  $$a \in cl(A)$$

  if $tp(a/A, M)$ is realized only countably many times in M.

Definition 5 *1. $M \in \mathcal{K}$ is* **prime** *over $A \subseteq M$, if for each $N \in \mathcal{K}$ containing A, there is an elementary map $f : M \to N$ which is the identity on A.*

2. $M \in \mathcal{K}$ is **minimal** *over $A \subseteq M$, if whenever $N \in \mathcal{K}$ contains A and $M \prec N$, then $M = N$.*

Second step

Proposition 2 *Let $\mathcal{K}$ be λ-categorical for some uncountable cardinal λ. Then*

- *Each uncountable model is prime and minimal over a quasi-minimal set.*

- *The dimension of any quasi-minimal set inside an uncountable model M is equal to the cardinality of M.*

Dimension theory beyond categoricity: Dividing and (Super)simplicity

Definition 6 *Let* $p(\bar{x}, \bar{c}) = tp(\bar{a}/B\bar{c})$.
We say that $p(\bar{x}, \bar{c})$ **divides over** B *if there exists an infinite* B*-indiscernible sequence* I *containing* $\bar{c}$ *such that*

$$\bigcup_{\bar{d} \in I} p(\bar{x}, \bar{d})$$

is not realized in $\mathfrak{C}$.

We write $A \underset{B}{\downarrow} C$ if $tp(\bar{a}/B\bar{c})$ does not divide over B, for every finite $\bar{a} \in A$ and finite $\bar{c} \in C$.

Definition 7 *We say that* $\mathfrak{C}$ *is* **supersimple** *if*
(1) **Local Character***:*
If $\bar{a}$ *is finite and* C *is a set, then there exists a finite* $B \subseteq C$ *such that* $\bar{a} \underset{B}{\downarrow} C$.
(2) **Extension***:*
Let $B \subseteq C$ *and* $\bar{a}$ *be given. There exists* $\bar{a}' \in \mathfrak{C}$ *realizing* $tp(\bar{a}/B, \mathfrak{C})$ *such that* $\bar{a}' \underset{B}{\downarrow} C$.

Algebraic Properties of Dividing

Theorem 3 (Buechler-Lessmann) *If $\mathfrak{C}$ is simple then*

- **Symmetry***: $A \underset{B}{\downarrow} C$ if and only if $C \underset{B}{\downarrow} A$.*

- **Transitivity***: Let $B \subseteq C \subseteq D$. If $A \underset{B}{\downarrow} C$ and $A \underset{C}{\downarrow} D$ then $A \underset{B}{\downarrow} D$.*

The key property is the use of Morley sequences, i.e. independent indiscernible sequences.

Although the context is more general, the proofs turn out to be simpler and shorter than in the first order case.

Multiplicity Properties: Chain Condition

Fix $A \subseteq \mathfrak{C}$. The set of types $tp(\bar{b}/A\bar{c})$, with

$$\bar{b} \underset{A}{\smile} \bar{c}, \quad \text{for finite sequences } \bar{b}, \bar{c} \in \mathfrak{C},$$

can be **partially ordered** under reverse inclusion. Two types $tp(\bar{b}_1/A\bar{c}_1)$ and $tp(\bar{b}_2/A\bar{c}_2)$ in this poset are **incompatible** if there does not exist a common realization $\bar{b} \in \mathfrak{C}$ such that $\bar{b} \underset{A}{\smile} \bar{c}_1\bar{c}_2$.

The following theorem asserts that this poset has a **chain condition** when $\mathfrak{C}$ is simple.

Theorem 4 (Buechler-Lessmann) *Let $\mathfrak{C}$ be a big homogeneous model. If $\mathfrak{C}$ is simple, then for each A, there exists a cardinal μ depending only on the size of A, such that if $\bar{b}_\ell \underset{A}{\smile} \bar{c}_\ell$ for $\ell < \mu$, then there exist $i < j < \mu$ and a sequence $\bar{b} \in \mathfrak{C}$ realizing $tp(\bar{b}_i/A\bar{c}_i) \cup tp(\bar{b}_j/A\bar{c}_j)$ such that*

$$\bar{b} \underset{A}{\smile} \bar{c}_i\bar{c}_j.$$

Stability and Simplicity

Definition 8 *A big homogeneous model $\mathfrak{C}$ is* **stable** *if there does not exist a formula $\phi(\bar{x};\bar{y})$ and a sequence $(\bar{a}_i : i < \|\mathfrak{C}\|)$ such that*

$$\mathfrak{C} \models \phi(a_i; a_j) \quad \textit{iff} \quad i < j.$$

Theorem 5 (Buechler-Lessmann) *Let $\mathfrak{C}$ be a big, simple, homogeneous model. The following conditions are equivalent:*

1. *$\mathfrak{C}$ is stable;*

2. **Bound on the number of free extensions***;*
   For each A, there exists a cardinal μ depending only on the size of A, such that if $\bar{b}_\ell \mathop{\smile\!\!\!\!\!\!|}_{A} C$ for $\ell < \mu$, then there exist $i < j < \mu$ such that

$$tp(\bar{b}_i/C) = tp(\bar{b}_j/C).$$

Concluding Remarks

Homogeneous model theory is at once general and well-behaved. It provides a powerful unifying framework for many mathematical contexts.

For example, linear dependence inside a vector space, algebraic dependence in field theory, algebraic dependence inside a strongly minimal structure are all examples of countable dependence inside a **quasi-minimal set**. More generally, all of these, including non-orthogonality in Hilbert spaces are particular cases of **dividing** inside a simple homogeneous structure. Hilbert spaces are simple and stable in this context, but not from a first order point of view, as they have the strict order property. More generally, Hilbert spaces with a self-adjoint operator, or a generic automorphism, or a generic predicate are simple and stable (Berenstein, Buechler-Berenstein). Also $(\ell^2(X), <, >, +, \times)$ is simple, stable (Berenstein).