ALGEBRAIC EXPANSIVE SUBDYNAMICS

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1. Expansive subdynamics

Let (X, ρ) be a compact metric space, which we assume is infinite unless otherwise stated. A $\mathbb{Z}^d$*-action* β on X is a homomorphism from $\mathbb{Z}^d$ to the group of homeomorphisms of X. For $\mathbf{n} \in \mathbb{Z}^d$ we denote the corresponding homeomorphism by $\beta^{\mathbf{n}}$, so that $\beta^{\mathbf{m}} \circ \beta^{\mathbf{n}} = \beta^{\mathbf{m}+\mathbf{n}}$, and $\beta^{\mathbf{0}}$ is the identity on X.

For a subset F of $\mathbb{R}^d$ put

$$\rho_\beta^F(x, y) = \sup\{\rho\big(\beta^{\mathbf{n}}(x), \beta^{\mathbf{n}}(y)\big) : \mathbf{n} \in F \cap \mathbb{Z}^d\},$$

and if $F \cap \mathbb{Z}^d = \varnothing$ define $\rho_\beta^F(x, y) = 0$.

Definition 1.1. A $\mathbb{Z}^d$-action β on (X, ρ) is *expansive* provided there is a $\delta > 0$ such that $\rho_\beta^{\mathbb{R}^d}(x, y) \le \delta$ implies that $x = y$. In this case δ is called an *expansive constant* for β.

Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^d$. For $F \subset \mathbb{R}^d$ and $\mathbf{x} \in \mathbb{R}^d$ define $\operatorname{dist}(\mathbf{x}, F) = \inf\{\|\mathbf{x} - \mathbf{y}\| : \mathbf{y} \in F\}$. For $t > 0$ put $F^t = \{\mathbf{x} \in \mathbb{R}^d : \operatorname{dist}(\mathbf{x}, F) \le t\}$, so that F^t is the result of thickening F by t.

Definition 1.2. Let β be a $\mathbb{Z}^d$-action on (X, ρ) and F be a subset of $\mathbb{R}^d$. Then F is *expansive for* β, or β is *expansive along* F, if there are $\varepsilon > 0$ and $t > 0$ such that $\rho_\beta^{F^t}(x, y) \le \varepsilon$ implies that $x = y$. If F fails to meet this condition it F is *nonexpansive for* β, or β is *nonexpansive along* F.

Next we examine subsets F that are linear subspaces of $\mathbb{R}^d$. Let $\mathsf{G}_k = \mathsf{G}_{d,k}$ denote the Grassmann manifold of k-dimensional subspaces (or simply k-*planes*) of $\mathbb{R}^d$. Recall that G_k is a compact manifold of dimension $k(d-k)$ whose topology is given by declaring two subspaces to be close if their intersections with the unit sphere are close in the Hausdorff metric.

Definition 1.3. For a $\mathbb{Z}^d$-action β define

$$\mathsf{E}_k(\beta) = \{V \in \mathsf{G}_k : V \text{ is expansive for } \beta\},$$
$$\mathsf{N}_k(\beta) = \{V \in \mathsf{G}_k : V \text{ is nonexpansive for } \beta\}.$$

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This is a joint work by Manfred Einsiedler, Douglas Lind, Richard Miles and Tom Ward.

Simple coding arguments [BL, Lemma 3.4] show that each $\mathsf{E}_k(\beta)$ is an open subset of G_k, so that each $\mathsf{N}_k(\beta)$ is compact. It is easy to see, if W is nonexpansive for β and V is a subspace of W, then V is also nonexpansive for β. A basic result [BL, Thm. 3.6] is the converse: If V is a nonexpansive subspace for β of dimension $\le d-2$, then there is a nonexpansive subspace for β containing V of one higher dimension. If X is infinite, then the zero subspace is nonexpansive, and hence inductively we see that each $\mathsf{N}_k(\beta) \neq \varnothing$ for $1 \le k \le d-1$. Furthermore, it follows that if $V \in \mathsf{N}_k(\beta)$, then there is a $W \in \mathsf{N}_{d-1}(\beta)$ that contains V. Hence $\mathsf{N}_k(\beta)$ consists of exactly all k-dimensional subspaces of the subspaces in $\mathsf{N}_{d-1}(\beta)$. Thus $\mathsf{N}_{d-1}(\beta)$ determines the entire expansive subdynamics of β.

In order to treat algebraic $\mathbb{Z}^d$-actions, it is convenient to shift our viewpoint slightly and use half-spaces in $\mathbb{R}^d$ rather than $(d-1)$-planes. Let $\mathsf{S}_{d-1} = \{\mathbf{v} \in \mathbb{R}^d : \|\mathbf{v}\| = 1\}$ be the unit $(d-1)$-sphere. For $\mathbf{v} \in \mathsf{S}_{d-1}$ define $H_{\mathbf{v}} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \cdot \mathbf{v} \le 0\}$ to be the half-space with outward unit normal $\mathbf{v}$. Let H_d be the set of half-spaces in $\mathbb{R}^d$, which we identify with S_{d-1} via the parameterization $\mathbf{v} \leftrightarrow H_{\mathbf{v}}$. For $H \in \mathsf{H}_d$ we denote its outward unit normal vector by $\mathbf{v}_H$.

Definition 1.4. For a $\mathbb{Z}^d$-action β define

$$\mathsf{E}(\beta) = \{H \in \mathsf{H}_d : H \text{ is expansive for } \beta\},$$
$$\mathsf{N}(\beta) = \{H \in \mathsf{H}_d : H \text{ is nonexpansive for } \beta\}.$$

Remark 1.5. A coding argument analogous to [BL, Lemma 3.4] shows that $\mathsf{E}(\beta)$ is an open set and so $\mathsf{N}(\beta)$ is a compact set.

The following lemma shows that a $(d-1)$-plane is nonexpansive for β if and only if at least one of the two bounding half-spaces is also nonexpansive for β. Thus if we define $\pi\colon \mathsf{H}_d \to \mathsf{G}_{d-1}$ by $\pi(H) = \partial H$, then $\pi(\mathsf{N}(\beta)) = \mathsf{N}_{d-1}(\beta)$. This shows that the half-space behavior $\mathsf{N}(\beta)$ determines the expansive subdynamics of β.

We start by recalling the following key notion from [BL, Definition 3.1].

Definition 1.6. Let β be an expansive $\mathbb{Z}^d$-action with expansive constant δ. For subsets E, F of $\mathbb{R}^d$ we say that E *codes* F provided that, for every $\mathbf{x} \in \mathbb{R}^d$, if $\rho_\beta^{E+\mathbf{x}}(x,y) \le \delta$ then $\rho_\beta^{F+\mathbf{x}}(x,y) \le \delta$.

Lemma 1.7. *Let β be a $\mathbb{Z}^d$-action and $V \in \mathsf{G}_{d-1}$. Then $V \in \mathsf{N}_{d-1}(\beta)$ if and only if there is an $H \in \mathsf{N}(\beta)$ with $\partial H = V$.*

Proof. If $H \in \mathsf{N}(\beta)$, then $V = \partial H \subset H$ is also nonexpansive.

Conversely, let $V \in \mathsf{G}_{d-1}$ and $H = H_{\mathbf{v}}$, $H' = H_{-\mathbf{v}}$ be the two half-spaces with boundary V. Suppose that both H and H' are expansive for β. We prove that V is also expansive for β, which will complete the proof.

Let δ be an expansive constant for β. Let $B(r)$ denote the ball of radius r in $\mathbb{R}^d$, and $[\mathbf{0}, \mathbf{v}]$ be the line segment joining $\mathbf{0}$ to $\mathbf{v}$. A "finite" version of the expansiveness of H, entirely analogous to [BL, Lemma 3.2], is that there is an $r > 0$ such that $H \cap B(r)$ codes $[\mathbf{0}, \mathbf{v}]$. Similarly, there is an $s > 0$ such that $H' \cap B(s)$ codes $[\mathbf{0}, -\mathbf{v}]$. Hence if $t = \max\{r, s\}$, then V^t codes V^{t+1}, which in turn codes V^{t+2}, and so on. Thus V^t codes $\mathbb{R}^d$, which means that V is expansive. $\square$

2. Algebraic $\mathbb{Z}^d$-actions

We just recall from the last talk the characterisation of expansiveness.

For an ideal $\mathfrak{a}$ in R_d put

$$\mathsf{V}(\mathfrak{a}) = \{\, \mathbf{z} = (z_1, \dots, z_d) \in (\mathbb{C}^\times)^d : f(z_1, \dots, z_d) = 0 \text{ for all } f \in \mathfrak{a}\}$$

(we omit 0 from $\mathbb{C}$ since we are using Laurent polynomials). Let $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$ and $\mathbb{S}^d = \{(z_1, \dots, z_d) \in \mathbb{C}^d : |z_1| = \cdots = |z_d| = 1\}$ be the multiplicative d-torus. Then we have the following characterization of expansiveness (see [S, Thm. 6.5]), which can the thought of as a generalization of the fact that a toral automorphism is expansive if and only if no eigenvalue lies on $\mathbb{S}$.

Theorem 2.1. *Let M be an R_d-module. Then the following are equivalent.*

(1) *α_M is expansive.*
(2) *M is Noetherian and $\alpha_{R_d/\mathfrak{p}}$ is expansive for every $\mathfrak{p} \in \operatorname{asc}(M)$.*
(3) *M is Noetherian and $\mathsf{V}(\mathfrak{p}) \cap \mathbb{S}^d = \varnothing$ for every $\mathfrak{p} \in \operatorname{asc}(M)$.*

Observe that $\rho\big(\alpha^{\mathbf{n}}(x), \alpha^{\mathbf{n}}(y)\big) = \rho\big(\alpha^{\mathbf{n}}(x - y), \alpha^{\mathbf{n}}(0_X)\big)$ since ρ is assumed to be translation-invariant. Thus in considering the expansive behavior of an algebraic $\mathbb{Z}^d$-action on a pair of points, we may assume that one of them is 0_X.

3. Characterization of expansive half-spaces

According to Theorem 2.1, there are two reasons that α_M may fail to be expansive: M may not be Noetherian, or there may be a point of $\mathsf{V}(\mathfrak{p}) \cap \mathbb{S}^d$ for some $\mathfrak{p} \in \operatorname{asc}(M)$.

Let H be a Halfspace, then we define $R_H = \mathbb{Z}[u^{\mathbf{n}} : \mathbf{n} \in H \cap \mathbb{Z}^d]$.

Theorem 3.1. *Let M be a Noetherian R_d-module, α_M be the corresponding algebraic $\mathbb{Z}^d$-action, and $H \in \mathsf{H}_d$. Then the following are equivalent.*

(1) *α_M is expansive along H.*
(2) *$\alpha_{R_d/\mathfrak{p}}$ is expansive along H for every $\mathfrak{p} \in \operatorname{asc}(M)$.*
(3) *$R_d/\mathfrak{p}$ is R_H-Noetherian and $[0, \infty)\mathbf{v}_H \cap \log|\mathsf{V}(\mathfrak{p})| = \varnothing$ for every $\mathfrak{p} \in \operatorname{asc}(M)$.*

Remark 3.2. Suppose that M is a Noetherian R_d-module, $H \in \mathsf{H}_d$, and $\mathbf{k} \in \mathbb{Z}^d \smallsetminus H$. Then one can show that if M is R_H-Noetherian then there is a polynomial of the form $u^{\mathbf{k}} - f(u)$ with $f(u) \in R_H$ and $\mathbf{k} \notin H$ which annihilates M. The existence of such a polynomial also implies that M is R_H-Noetherian, so this provides a convenient criterion for a Noetherian R_d-module to be R_H-Noetherian.

The content of Theorem 3.1 can be summarized by the equalities

$$\mathsf{E}(\alpha_M) = \bigcap_{\mathfrak{p}\in\mathrm{asc}(M)} \mathsf{E}(\alpha_{R_d/\mathfrak{p}}), \quad \mathsf{N}(\alpha_M) = \bigcup_{\mathfrak{p}\in\mathrm{asc}(M)} \mathsf{N}(\alpha_{R_d/\mathfrak{p}}).$$

4. Examples

Our examples involve three or fewer variables, so for notational simplicity we use u, v, w, instead of u_1, u_2, u_3. Using the correspondence $\mathsf{H}_d \leftrightarrow \mathsf{S}_{d-1}$ given by $H \leftrightarrow \mathbf{v}_H$, we identify subsets of H_d with the corresponding subsets of S_{d-1} for ease of visualization. For an ideal $\mathfrak{a} \in R_d$ put

$$\mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\mathfrak{a}}) = \{\mathbf{v} \in \mathsf{S}_{d-1} : R_d/\mathfrak{a} \text{ is not } R_{H_{\mathbf{v}}}\text{-Noetherian}\},$$
$$\mathsf{N}^{\mathsf{v}}(\alpha_{R_d/\mathfrak{a}}) = \{\mathbf{v} \in \mathsf{S}_{d-1} : [0,\infty)\mathbf{v} \cap \log|\mathsf{V}(\mathfrak{a})| \neq \varnothing\}.$$

By Theorem 3.1,

$$\mathsf{N}(\alpha_{R_d/\mathfrak{a}}) = \mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\mathfrak{a}}) \cup \mathsf{N}^{\mathsf{v}}(\alpha_{R_d/\mathfrak{a}}).$$

For a principal ideal $\langle f\rangle$ in R_d we abbreviate $\mathsf{V}(\langle f\rangle)$ to $\mathsf{V}(f)$.

Example 4.1. (*One variable, principal ideal.*) Let $0 \neq f(u) \in R_1$, which we assume is not a unit in R_1 (i.e., not ± 1 times a monomial). Multiplying $f(u)$ by a monomial if necessary, we can also assume that $f(u) = c_r u^r + c_{r-1}u^{r-1} + \cdots + c_1 u + c_0$, where $c_j \in \mathbb{Z}$ and $c_r c_0 \neq 0$. The unit "sphere" in $\mathbb{R}$ is $\mathsf{S}_0 = \{1, -1\}$. By Remark 3.2, $1 \in \mathsf{N}^{\mathsf{n}}(\alpha_{R_1/\langle f\rangle})$ if and only if $|c_r| > 1$. If $|c_r| = 1$, then $f(u) = \pm\prod_{j=1}^{r}(u - \lambda_j)$, so that $\prod_{j=1}^{r}|\lambda_j| = |c_0| \geq 1$. Hence some $\lambda_k \in \mathsf{V}(f)$ has $|\lambda_k| \geq 1$. If $|\lambda_k| = 1$, the $\alpha_{R_1/\langle f\rangle}$ is not expansive, and so trivially $1 \in \mathsf{N}(\alpha_{R_1/\langle f\rangle})$. If $|\lambda_k| > 1$, then $[0,\infty)\cdot 1 \cap \log|\mathsf{V}(f)| \neq \varnothing$, and so $1 \in \mathsf{N}^{\mathsf{v}}(\alpha_{R_1/\langle f\rangle})$. In all cases we conclude that $1 \in \mathsf{N}(\alpha_{R_1/\langle f\rangle})$. Similarly, $-1 \in \mathsf{N}(\alpha_{R_1/\langle f\rangle})$. Thus $\mathsf{N}(\alpha_{R_1/\langle f\rangle}) = \mathsf{S}_0$.

The following polynomials illustrate some possible combinations of N^{n} and N^{v}.

(a) $f(u) = u^2 - u - 1$, $\mathsf{N}^{\mathsf{n}} = \varnothing$ and $\mathsf{N}^{\mathsf{v}} = \{1, -1\}$.
(b) $f(u) = u - 2$, $\mathsf{N}^{\mathsf{n}} = \{-1\}$ and $\mathsf{N}^{\mathsf{v}} = \{1\}$.
(c) $f(u) = 2$, $\mathsf{N}^{\mathsf{n}} = \{1, -1\}$ and $\mathsf{N}^{\mathsf{v}} = \varnothing$.
(d) $f(u) = 2u^2 - 6u + 3$, $\mathsf{N}^{\mathsf{n}} = \{1, -1\}$ and $\mathsf{N}^{\mathsf{v}} = \{1, -1\}$.

For the principal ideals above we found that the nonexpansive set is the whole sphere. This is true in general.

Proposition 4.2. *Let $f \in R_d$ generate a proper ideal. Then* $\mathsf{N}(\alpha_{R_d/\langle f\rangle}) = \mathsf{S}_{d-1}$.

For nonprincipal prime ideals the nonexpansive set can exhibit more variety.

Example 4.3. (*Ledrappier's example.*) Let $\mathfrak{p} = \langle 2, 1 + u + v\rangle$, which is easily seen to be a prime ideal in R_2. Then $\mathsf{V}(\mathfrak{p}) = \varnothing$, so that $\mathsf{N}^{\mathsf{v}}(\alpha_{R_2/\mathfrak{p}}) = \varnothing$. On the other hand, the outward normals to the sides of the Newton polygon $\mathcal{N}(1+u+v)$ are nonexpansive, and by Remark 3.2 these are the only nonexpansive vectors. Thus $\mathsf{N}(\alpha_{R_2/\mathfrak{p}}) = \{\mathbf{v}_{\pi/4}, \mathbf{v}_{\pi}, \mathbf{v}_{3\pi/2}\}$.

Example 4.4. (*Three-dimensional Ledrappier example.*) Let $\mathfrak{p} = \langle 2, 1+u+v+w\rangle$, again easily seen to be a prime ideal in R_3. Then $\mathsf{V}(\mathfrak{p}) = \varnothing$, so that $\mathsf{N}^{\mathsf{v}}(\alpha_{R_3/\mathfrak{p}}) = \varnothing$. The nonexpansive set was determined in [BL, Example 2.9] to be the 1-skeleton of the spherical dual to the Newton polytope $\mathcal{N}(1+u+v+w)$. This is depicted in Figure 1, where $\mathsf{N}^{\mathsf{n}}(\alpha_{R_3/\mathfrak{p}})$ consists of the six great circle arcs determined by the six edges of $\mathcal{N}(1+u+v+w)$.

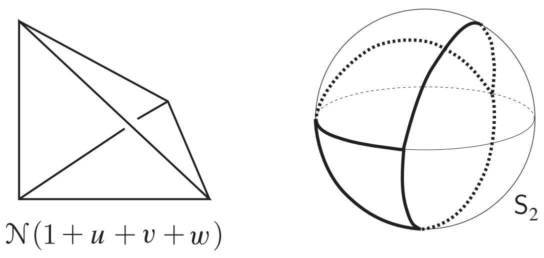


FIGURE 1. The nonexpansive set for three-dimensional Ledrappier

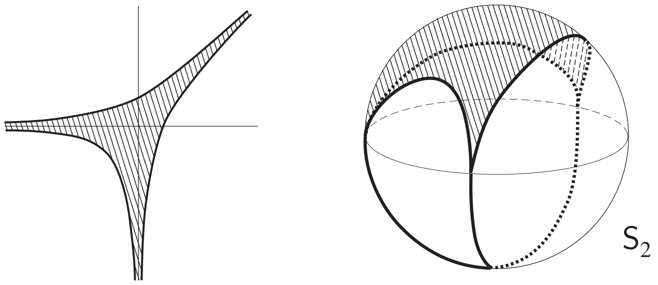
Example 4.5. Let $\mathfrak{p} = \langle u-2, v-3\rangle$. Then $\mathsf{V}(\mathfrak{p})$ consists of one point, so there is only the hyperplane with normal vector $(\log 2, \log 3)$, which is not expansive due to the variety condition. It is easy to see that every hyperplane with the exception of two are Noetherian. The execptions have the normal vectors $(0,-1)$ and $(-1,0)$. The first corresponds to the $\mathbb{Z}[1/6]$ viewed as a module over $\mathbb{Z}[1/2]$, so $H_{(0,-1)} \in \mathsf{N}^{\mathsf{n}}$. Therefore

$$\begin{aligned} \mathsf{N}^{\mathsf{v}} &= \{H_{(\log 2, \log 3)}\} \\ \mathsf{N}^{\mathsf{n}} &= \{H_{(0,-1)}, H_{(-1,0)}\} \end{aligned}$$

Example 4.6. (*Nonexpansive set with nonempty interior.*) Let $d = 3$ and $\mathfrak{p} = \langle 1+u+v, w-2\rangle$.

We determine $\mathsf{N}^{\mathsf{v}}(\alpha_{R_3/\mathfrak{p}})$. Since $\mathsf{V}(\mathfrak{p}) = \{(z, -z-1, 2) : z \in \mathbb{C}\}$, we see that $\log|\mathsf{V}(\mathfrak{p})|$ lies in a plane at height $\log\ 2$ above the origin, and in this plane it has the shape shown in Figure 2(a), where the boundary curves are parameterized by $(\log r, \log|r \pm 1|)$ for $0 < r < \infty$. When projected radially to S_2, we obtain the set in the upper hemisphere shown in Figure 2(b), with three cusps on the equator.

Using Remark 3.2, the polynomial $w - 2 \in \mathfrak{p}$ shows that the open upper hemisphere in S_2 is disjoint from $\mathsf{N}^{\mathsf{n}}(\alpha_{R_3/\mathfrak{p}})$. Furthermore, $1+u+v \in \mathfrak{p}$ shows that no points in the lower hemisphere are in $\mathsf{N}^{\mathsf{n}}(\alpha_{R_3/\mathfrak{p}})$ either, with the possible exceptions of those on the three quarter meridians shown in Figure 2(b). One can show that each of these quarter meridians is contained in $\mathsf{N}^{\mathsf{n}}(\alpha_{R_3/\mathfrak{p}})$, so that they, combined with $\mathsf{N}^{\mathsf{v}}(\alpha_{R_3/\mathfrak{p}})$ in the upper hemisphere, comprise all of $\mathsf{N}(\alpha_{R_3/\mathfrak{p}})$.



(a) (b)

FIGURE 2. Nonexpansive set for Example 4.6

Remark 4.7. In the previous example, and in many others, the non-Noetherian and variety parts of the nonexpansive set are "glued together" along a set which can be described by the asymptotic behavior of the logarithmic image of the variety. For instance, in Example 4.6, the variety part $\mathsf{N}^{\mathsf{v}}(\alpha_{R_3/\mathfrak{p}})$ in the upper hemisphere and the non-Noetherian part $\mathsf{N}^{\mathsf{n}}(\alpha_{R_3/\mathfrak{p}})$ in the lower hemisphere are glued together at the three cusp points on the equator, which are also the three asymptotic directions of $\log|\mathsf{V}(\mathfrak{p})|$. This illustrates a general phenomenon of the *logarithmic limit set of an algebraic variety*, introduced by Bergman [B]. He showed that this set is always contained in a finite union of lower dimensional great spheres. We point out that this set has an alternative description as the set of half-spaces H for which $R_d/\mathfrak{p}\otimes\mathbb{Q}$ is not Noetherian over $R_H\otimes\mathbb{Q}$, so in this sense is also a limiting part of the non-Noetherian set (we are grateful to Bernd Sturmfels for pointing this out to us).

5. FURTHER ANALYSIS OF NONEXPANSIVE SETS

Let M be an R_d-module, which we will assume throughout this section to be Noetherian. In §3 we showed that

$$\mathsf{N}(\alpha_M) = \bigcup_{\mathfrak{p}\in\operatorname{asc}(M)} \mathsf{N}(\alpha_{R_d/\mathfrak{p}}), \tag{5.1}$$

where

$$\mathsf{N}(\alpha_{R_d/\mathfrak{p}}) = \mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\mathfrak{p}}) \cup \mathsf{N}^{\mathsf{v}}(\alpha_{R_d/\mathfrak{p}}). \tag{5.2}$$

We begin by extending the definitions of N^{n} and N^{v}.

Definition 5.1. For M a Noetherian R_d-module, define

$$\mathsf{N}^{\mathsf{n}}(\alpha_M) = \bigcup_{\mathfrak{p}\in\operatorname{asc}(M)} \mathsf{N}^{\mathsf{n}}(\alpha_{R_d/\mathfrak{p}}) \quad\text{and}\quad \mathsf{N}^{\mathsf{v}}(\alpha_M) = \bigcup_{\mathfrak{p}\in\operatorname{asc}(M)} \mathsf{N}^{\mathsf{v}}(\alpha_{R_d/\mathfrak{p}}). \tag{5.3}$$

Hence by (5.1) and (5.2), $\mathsf{N}(\alpha_M) = \mathsf{N}^{\mathsf{n}}(\alpha_M)\cup\mathsf{N}^{\mathsf{v}}(\alpha_M)$.

Remark 5.2. Alternatively, we could have defined $\mathsf{N}^{\mathsf{n}}(\alpha_M)$ to be $\{H \in \mathsf{H}_d : M \text{ is not } R_H\text{-Noetherian}\}$. For M has a prime filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ with $M_j/M_{j-1} \cong R_d/\mathfrak{q}_j$, where $\mathfrak{q}_j$ is a prime ideal containing some $\mathfrak{p}_j \in \operatorname{asc}(M)$, and furthermore every $\mathfrak{p} \in \operatorname{asc}(M)$ occurs as some $\mathfrak{q}_j$ [E, p. 93]. Then M fails to be R_H-Noetherian if and only if some quotient M_j/M_{j-1} is not R_H-Noetherian, establishing the equivalence of the two definitions of $\mathsf{N}^{\mathsf{n}}(\alpha_M)$.

Next, we characterize when $\mathsf{N}^{\mathsf{n}}(\alpha_M)$ or $\mathsf{N}^{\mathsf{v}}(\alpha_M)$ is empty. Note that the answers depend only on the topological nature of X_M. If $\mathfrak{p}$ is a prime ideal in R_d, we let $\operatorname{char}(R_d/\mathfrak{p})$ denote the characteristic of the integral domain $R_d/\mathfrak{p}$.

Proposition 5.3. *Let M be a Noetherian R_d-module.*

(1) *The following conditions are equivalent:*
   (a) $\mathsf{N}^{\mathsf{n}}(\alpha_M) = \varnothing$.
   (b) *M is finitely generated as an abelian group.*
   (c) *X_M is the direct product of a finite-dimensional torus and a finite abelian group.*

(2) *The following conditions are equivalent:*
   (1) $\mathsf{N}^{\mathsf{v}}(\alpha_M) = \varnothing$.
   (2) *M is a torsion abelian group.*
   (3) *X_M is totally disconnected.*

Although a $\mathbb{Z}^d$-action nominally involves d commuting transformations, sometimes they satisfy a lot of relation so that for instance for entropy one should look at subdynamics of a certain dimension. We next turn to ways of measuring this rank. In what follows h denotes topological entropy.

Definition 5.4. Let β be a topological $\mathbb{Z}^d$-action. Define the *expansive rank* of β to be

$$\operatorname{exprk}(\beta) = \min\{k : \mathsf{E}_k(\beta) \neq \varnothing\},$$

and the *entropy rank* of β to be

$$\operatorname{entrk}(\beta) = \max\{k : \text{there is a rational } k\text{-plane } V \text{ with } h(\beta, V \cap \mathbb{Z}^d) > 0\}.$$

By convention, if β is not expansive we put $\operatorname{exprk}(\beta) = d+1$, and if the set defining $\operatorname{entrk}(\beta)$ is empty we put $\operatorname{entrk}(\beta) = 0$.

The building blocks for algebraic $\mathbb{Z}^d$-actions are based on modules of the form $R_d/\mathfrak{p}$. For these we investigate expansive and entropy ranks. The appropriate algebraic version of rank is algebraic dimension, in this case Krull dimension for rings. Recall that the Krull dimension $\operatorname{kdim} \mathcal{R}$ of a ring $\mathcal{R}$ is the length r of the longest chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_r$ of prime ideals in $\mathcal{R}$ (see [E, Chapter 8] for the necessary background). For example, $\operatorname{kdim} R_d = d+1$. In [BL, Theorem 7.5] it is shown that if $\mathfrak{p}$ is a prime ideal in R_d generated by g elements, then

$$\operatorname{exprk}(\alpha_{R_d/\mathfrak{p}}) \geq \operatorname{kdim}(R_d/\mathfrak{p}) - 1 \geq d - g,$$

and if $\mathrm{char}(R_d/\mathfrak{p}) > 0$ then

$$\mathrm{exprk}(\alpha_{R_d/\mathfrak{p}}) \geq d - g + 1.$$

Since expansive and entropy ranks in some sense measure dimension, one might heuristically expect $\mathrm{exprk}(\alpha_{R_d/\mathfrak{p}})$, $\mathrm{entrk}(\alpha_{R_d/\mathfrak{p}})$ and $\mathrm{kdim}(R_d/\mathfrak{p})$ to coincide. The best that can be said in this direction is as follows.

Proposition 5.5. *Let $\alpha_{R_d/\mathfrak{p}}$ be an expansive action with zero entropy. Then*

$$\mathrm{entrk}(\alpha_{R_d/\mathfrak{p}}) = \mathrm{kdim}(R_d/\mathfrak{p}) \leq \mathrm{exprk}(\alpha_{R_d/\mathfrak{p}}).$$

If $\mathrm{char}(R_d/\mathfrak{p}) > 0$ *then all of these quantities are equal.*

Example 5.6. Despite to the hope of a better result in Proposition 5.5 there is an example of an algebraic $\mathbb{Z}^3$-action corresponding to a module of the form $R_3/\mathfrak{p}$. This example is somehow a glued version of Example 4.6 so that N^v covers everything. This shows that the inequality in Proposition 5.5 can be strict.

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