On measures on convex Corson compacta

Grzegorz Plebanek

(University of Wroclaw)

Given κ, the Corson space $\Sigma(R^\kappa)$ is

$$\Sigma(R^\kappa) = \{x \in R^\kappa : \ |\{\alpha < \kappa : x(\alpha) \neq 0\}| \leq \omega\}.$$

A *Corson compactum* is any space which is homeomorphic to a compact subset of $\Sigma(R^\kappa)$ for some κ.

Remark. *A Corson compact K is separable if and only if K is metrizable.*

$P(K)$ denotes the set of all probability Radon measures on K. Every $\mu \in P(K)$ is a Borel measure on K, such that for any Borel set B

$$\mu(B) = \sup\{\mu(F) : \ F \subseteq B, \ F \text{ compact}\}.$$

$P(K)$ is equipped with the $weak^*$ topology inherited from $C(K)^*$. Basic neighbourhoods of $\mu \in P(K)$ are of the form

$$\{\nu \in P(K) : \ |\int g_i \,\mathrm{d}\mu - \int g_i \,\mathrm{d}\nu| < \varepsilon \text{ for } i \leq k\}.$$

A measure μ is called *separable* or of *countable type* if $L_1(\mu)$ is a separable Banach space.

Theorem. *If $MA(\omega_1)$ holds then every Radon measure on a Corson compact space has a metrizable support.*

Fact. Let K be a compact convex subset of $\Sigma(R^\kappa)$. For $\alpha < \kappa$, π_α is the usual projection. Every $\mu \in P(K)$ has a barycentre $b_\mu \in K$, i.e.

$$\int \pi_\alpha \, \mathrm{d}\mu = \pi_\alpha(b_\mu) = b_\mu(\alpha).$$

Theorem Let K be a compact convex subset of $\Sigma(R^\kappa)$.

(1) If $\mu \in P(K)$ is of countable type then μ has a metrizable support.

(2) If K is included in $\Sigma(R^\kappa_+)$ then every $\mu \in P(K)$ has a metrizable support.

Problem. (O. Kalenda) Can we prove in ZFC that every $\mu \in P(K)$ has a metrizable suupport whenever K is compact and convex subset of $\Sigma(R^\kappa)$?

Theorem 1. **[CH]** *There are a non-separable 0–dimensional Corson compact space K, a strictly positive measure $\mu \in P(K)$ of uncountable type, and a family $(g_\alpha)_{\alpha<\omega_1} \subseteq C(K)$ of norm one functions such that for every $\nu \in P(K)$, the set*

$$\{\xi < \omega_1 : \int g_\xi \, \mathrm{d}\nu \neq 0\},$$

is countable.

Theorem 2. *A compact space K has a strictly positive Radon measure if and only if the space $(P(K), weak^*)$ has a strictly positive Radon measure.*

Theorem 3. **[CH]** *There exists a compact convex nonseparable subspace L of $\Sigma(R^{\omega_1})$, and a strictly positive measure $\lambda \in P(L)$ (which has no metrizable support).*

$$\Phi(\nu) = (\int g_\xi \, \mathrm{d}\nu)_{\xi<\omega_1}$$

$$\Phi : P(K) \to \Sigma(R^{\omega_1})$$

Φ is an affine continuous mapping, and hence $L = \Phi(P(K))$ is a compact convex subset of $\Sigma(R^{\omega_1})$.

$P(K)$ has a strictly positive Radon measure Υ by Theorem 2. The measure $\lambda = \Phi(\Upsilon)$ is strictly positive on L.

Claim. There is an algebra $\mathcal{A} \subseteq Ba\{0,1\}^{\omega_1}$ such that

1. $\mathcal{A} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$,

   $|\mathcal{A}_\alpha| = \omega$, $\mathcal{A}_\beta \subseteq \mathcal{A}_\alpha$ for $\beta < \alpha < \omega_1$,

   $\mathcal{A}_0$ is some nonatomic algebra;

2. $\lambda(A) > 0$ for $\emptyset \neq A \in \mathcal{A}$;

3. $\mathcal{A}_\alpha = alg(\bigcup_{\beta<\alpha} \mathcal{A}_\beta \cup \{H_\alpha^0, H_\alpha^1\})$,

   $H_\alpha^0 \cap H_\alpha^1 = \emptyset$, $F_\alpha := H_\alpha^0 \cup H_\alpha^1 \neq \emptyset$,

   and for every $\beta < \alpha$ and $C \in \mathcal{A}_\beta$

   $\lambda(C \cap H_\alpha^0) = \lambda(C \cap H_\alpha^1)$;

4. if $(A_n) \subseteq \mathcal{A}$ is decreasing and

   $\lim \lambda(A_n) = 0$

   then there is $\alpha < \omega_1$ such that for all $\eta > \alpha$ we have $A_n \cap F_\eta = \emptyset$ for large n.

S. Argyros, S. Negrepontis, *Chain conditions for spaces of measues*, Preprint (1982).

S. Argyros, S. Mercourakis, S. Negrepontis, *Functional analityc properties of Corson–compact spaces*, Studia Math. 89 (1988) 197-228.

R. Haydon, *On dual L^1–spaces and injective bidual Banach spaces*, Israel J. Math. 31 (1978) 142–152.

K. Kunen, *A compact L–space under CH*, Topology Appl. 12 (1981), 283–287.

K. Kunen, J. van Mill, *Measures on Corson compact spaces*, Fund. Math. 147 (1995), 61–72.

G. Plebanek, *Nonseparable Radon measures and small compact spaces*, Fund. Math. 153 (1997), 25–40.

S. Todorcevic, *Chain condition methods in topology*, Topology Appl. 101 (2000), 45–82.