AN INTERESTING APPLICATION OF CLASSICAL TOPOLOGY IN REAL ANALYSIS

Dušan Repovš

Institute of Mathematics

Physics and Mechanics

Jadranska 19, P. O. Box 2964

Ljubljana, Slovenia 1001

e-mail: dusan.repovs@fmf.uni-lj.si

1. Introduction

Recall the ε-δ definition of continuity of a map $f : (X, d) \to (Y, \rho)$ between metric spaces:

$$(\forall x \in X)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x' \in X)$$

$$d(x, x') < \delta \Longrightarrow \rho(f(x), f(x')) < \varepsilon$$

i.e. there exists a (*not* necessarily continuous) function $\delta : X \times \mathcal{C}(X, Y) \times \mathbb{R}_+^* \to \mathbb{R}_+^*$ such that

$$d(x, x') < \delta(x, f, \varepsilon) \Longrightarrow \rho(f(x), f(x')) < \varepsilon,$$

for all values of variables

$$x, x' \in X, \ f \in \mathcal{C}(X, Y) \text{ and } \varepsilon \in \mathbb{R}_+^*.$$

Here $\mathcal{C}(X, Y)$ denotes the space of all continuous maps from X to Y, endowed with the following metric of *uniform convergence*:

$$\rho_T(f, g) = \sup\{\min\{1, \rho(f(x), g(x))\} | x \in X\}.$$

Let each such function δ be called a *modulus of continuity.* The following was the first result on existence of *continuous* moduli of continuity:

Theorem 1 (Repovš–Semenov) *Let* $f : (X,d) \to (Y,\rho)$ *be a continuous map between metric spaces and* X *a locally compact space. Then there exists a continuous function* $\hat{\delta} : X \times \mathcal{C}(X,Y) \times \mathbb{R}^*_+ \to \mathbb{R}^*_+$ *such that for every* $(x,f,\varepsilon) \in X \times \mathcal{C}(X,Y) \times \mathbb{R}^*_+$ *and every* $x' \in X$ *the following holds:*

$$d(x,x') < \hat{\delta}(x,f,\varepsilon) \Longrightarrow \rho(f(x),f(x')) < \varepsilon.$$

Consider two elementary applications of this theorem. First, recall two classical results from analysis: Let $f : X \to \mathbb{R}$ be a continuous function on a *compact* metric space X.

Then the *Weierstrass theorem* asserts that f is bounded (above and below) and the *Cantor theorem* asserts that f is uniformly continuous on X.

Corollary 1 *The Cantor theorem is a corollary of the Weierstrass theorem.*

Proof. Suppose that $f : (X, d) \to \mathbb{R}$ is continuous and X compact. Apply Theorem 1 to get a continuous modulus of continuity $\hat{\delta} : X \times \mathcal{C}(X, Y) \times \mathbb{R}_+^* \to \mathbb{R}_+^*$.

For a fixed $f \in \mathcal{C}(X, Y)$ and $\varepsilon \in \mathbb{R}_+^*$ we can now choose $\delta > 0$ as follows

$$\delta = \min\{\hat{\delta}(x, f, \varepsilon) | x \in X\},$$

since by the Weierstrass theorem such minimum exists. Hence f is uniformly continuous.

□

Corollary 2 *Let $\mathcal{G} \subset \mathcal{C}(X,Y)$ be a subset such that its closure $Cl(\mathcal{G})$ is compact. Then $\mathcal{G}$ is an equicontinous set of maps.*

Proof. Recall that $\mathcal{G}$ is said to be *equicontinuous* if for each $\varepsilon \in \mathbb{R}_+^*$ there exists $\delta \in \mathbb{R}_+^*$ such that for every $f \in \mathcal{G}$ and $x, x' \in X$

$$d(x,x') < \delta \Longrightarrow \rho(f(x), f(x')) < \varepsilon.$$

So under the hypothesis, it suffices to pick, for a fixed $\varepsilon \in \mathbb{R}_+^*$, our $\delta > 0$ as follows

$$\delta = \min\{\hat{\delta}(x,f,\varepsilon) | x \in X, f \in \mathrm{Cl}(\mathcal{G})\}.$$

$\square$

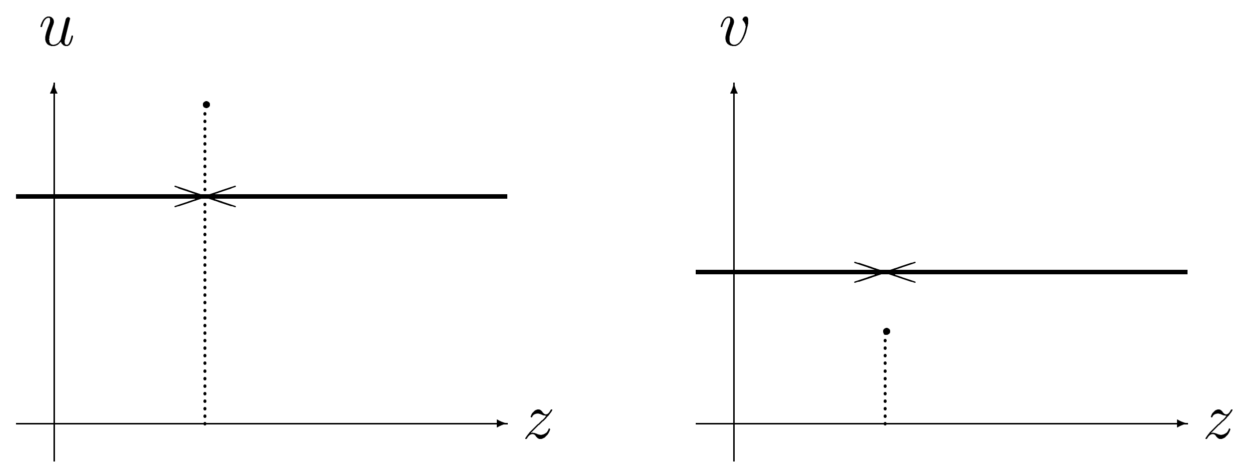
2. Alternative proofs of Theorem 1

A map $u : Z \to \mathbb{R}$ is called *upper* (resp. *lower*) *semicontinuous* at $z \in Z$ if for every $\varepsilon > 0$ there exists a neighbourhood $U \subset Z$ of z such that $u(z') < u(z) + \varepsilon$ (resp. $u(z') > u(z) - \varepsilon$), for each $z' \in U$.

Example. The maps $u, v : \mathbb{R} \to \mathbb{R}$ given by

$$u(z) = \begin{cases} 4; \text{ if } & z = 2 \\ 3; \text{ if } & z \neq 2 \end{cases} \qquad v(z) = \begin{cases} 1; \text{ if } & z = 2 \\ 2; \text{ if } & z \neq 2 \end{cases}$$

are upper (resp.) lower semicontinuous.



Theorem 2 (Dowker) *Let $u, v : Z \to \mathbb{R}$ be any functions, defined on a paracompact space Z, such that $u(z) > v(z)$, for every $z \in Z$. Assume also that v is upper semicontinuous and u is lower semicontinuous. Then there exists a continuous $w : Z \to \mathbb{R}$ such that $u(z) > w(z) > v(z)$, for every $z \in Z$.*

Proof of Theorem 1(using Dowker's theorem). Denote by $B_\delta(x)$ the open ball of radius δ with center $x \in X$. Introduce a function $\Delta : X \times \mathcal{C}(X, Y) \times \mathbb{R}_+^* \to \mathbb{R}$ as follows

$\Delta(x, f, \varepsilon) = \sup\{\delta > 0 | \mathrm{Cl}(B_\delta(x))$ is a compact set and $d(x, x') < \delta \Longrightarrow \rho(f(x), f(x')) < \varepsilon\}$, for each $x' \in X$.

By local compactness of X, Δ is well-defined and lower semicontinuous.

Apply Dowker's Theorem to

$$\Delta : X \times \mathcal{C}(X, Y) \times \mathbb{R}_+^* \to \mathbb{R}$$

and the zero-function

$$X \times \mathcal{C}(X, Y) \times \mathbb{R}_+^* \to \{0\}$$

to obtain a continuous function

$$\hat{\delta} : X \times \mathcal{C}(X, Y) \times \mathbb{R}_+^* \to \mathbb{R}_+^*$$

such that $\Delta(x, f, \varepsilon) > \hat{\delta}(x, f, \varepsilon) > 0$.

$\square$

A *multivalued* map $F : X \to Y$ is said to be *lower semicontinuous* if for every open subset $G \subset Y$ the set $\{x \in X | F(x) \cap G \neq \emptyset\}$ is open in X. A *singlevalued* map $f : X \to Y$ is called a *selection* of a multivalued map $F : X \to Y$ if $f(x) \in F(x)$, for each $x \in X$.

A convex subset of a Banach space is said to be of *D-type* if it contains all interior (in the convex sense) points of its closure. A point of a closed convex subset of a Banach space is said to be *interior* (in the convex sense) if it is not contained in any supporting hyperspace. It is easy to see that all finite-dimensional convex sets are examples of convex D-type sets. In this case the set of all interior (in the convex sense) points coincides with the set of all interior (in the usual sense) points.

Theorem 3 (Michael) *Let Z be a perfectly normal space and B a separable Banach space. Then each lower semicontinuous map $F : Z \to B$ with convex D-type images has a continuous selection $f : Z \to B$.*

Proof of Theorem 1(using Michael's theorem). Let $F : X \times \mathcal{C}(X, Y) \times \mathbb{R}_+^* \to \mathbb{R}_+^*$ be a multivalued map, given by $F(x, f, \varepsilon) = (0, \Delta(x, f, \varepsilon)]$, where Δ was defined earlier. Then F is lower semicontinuous since Δ is lower semicontinuous. Furthermore, each interval is a convex D-type set and the space $X \times \mathcal{C}(X, Y) \times \mathbb{R}_+^*$ is perfectly normal since it is metrizable. Apply Michael's Theorem to get a continuous selection $\hat{\delta} : X \times \mathcal{C}(X, Y) \times \mathbb{R}_+^* \to \mathbb{R}_+^*$ such that $\hat{\delta}(x, f, \varepsilon) \in (0, \Delta(x, f, \varepsilon)]$.

$\square$

3. Continuity-like predicates

Proof of Theorem 1 (without the hypothesis of local compactness of X, due to Artico and Marconi). Introduce the sets:

$$C^{\text{true}} = \{(x, x', f, \varepsilon) \in X \times X \times \mathcal{C}(X, Y) \times \mathbb{R}_+^* | \\ \rho(f(x), f(x')) < \varepsilon\}$$

$$C^{\text{false}} = \{(x, x', f, \varepsilon) \in X \times X \times \mathcal{C}(X, Y) \times \mathbb{R}_+^* | \\ \rho(f(x), f(x')) \geq \varepsilon\}.$$

The function $(x, x', f, \varepsilon) \mapsto \rho(f(x), f(x')) - \varepsilon$ is continuous, hence the set C^{true} is open and thus C^{false} is closed.

Obviously $(x, x, f, \varepsilon) \in C^{\text{true}}$. Therefore, the function $\hat{\delta} : X \times \mathcal{C}(X, Y) \times \mathbb{R}_+^* \to \mathbb{R}_+$ defined by: $\hat{\delta}(x, f, \varepsilon) = \text{dist}((x, x, f, \varepsilon), C^{\text{false}})$ is strictly positive. It remains to verify that $\hat{\delta}$ is continuous.

This proof gives an idea how to generalize Theorem 1. Let $\mathcal{F}(X,Y)$ be the space of all (not necessarily continuous) maps, equipped with the norm of uniform convergence. Let

$$P : X \times X \times \mathcal{F}(X,Y) \times \mathbb{R}^*_+ \to \{\text{true}, \text{false}\}$$

be a logical predicate and let $P^{\text{true}} = P^{-1}(\text{true})$ and $P^{\text{false}} = P^{-1}(\text{false})$.

Definition 1 If for each $x \in X$ and $\varepsilon \in \mathbb{R}^*_+$ there exists a neighbourhood $U \subset X$ of x such that $(x, x', f, \varepsilon) \in P^{\text{true}}$, for all $x' \in U$ then f is called *P-continuous.* If there exists at least one P-continuous map, then P is called a *continuity-like* predicate. Let $\mathcal{F}_P(X,Y) \subset \mathcal{F}(X,Y)$ be the subspace of all P-continuous maps and $P_0^{\text{true}} = P^{\text{true}} \cap (X \times X \times \mathcal{F}_P \times \mathbb{R}^+_*)$.

Example. Define the predicate C by :

$$C(x, x', f, \varepsilon) = (\rho(f(x), f(x')) < \varepsilon),$$

where $f : X \to Y$ is a map from a topological space X into a metric space (Y, ρ). Then C-continuous maps are precisely the continuous maps, hence C is the predicate of the usual continuity.

Example. Define the predicate USC by:

$$USC(x, x', f, \varepsilon) = (f(x') < f(x) + \varepsilon),$$

where $f : X \to \mathbb{R}$ is a function defined on a topological space X. Then USC-maps coincides with the upper semicontinuous maps, hence USC is the predicate of upper semicontinuity.

Example. Analogously, define the predicate LSC by:

$$LSC(x, x', f, \varepsilon) = (f(x') > f(x) - \varepsilon),$$

where $f : X \to \mathbb{R}$ is a function defined on a topological space X. Then LSC is the predicate of lower semicontinuity.

Example. Let $\alpha : X \to [0, \infty]$ be any function. Define the predicate C^{α} by:

$$C^{\alpha}(x, x', f, \varepsilon) = (\rho(f(x), f(x')) < \alpha(x) + \varepsilon),$$

where $f : X \to Y$ is a map from a topological space X into a metric space (Y, ρ).

The predicate C^{α} induces the notion of α-continuity, which is a generalization of the usual continuity, i.e. each continuous map is α-continuous for arbitrary α.

Moreover, if $\alpha_1(x) > \alpha_2(x)$, for all $x \in X$ then each α_2-continuous map is also α_1-continuous. For example, 0-continuity is the usual continuity and ∞-continuity is a void condition, i.e. each map is ∞-continuous.

Remark. All predicates listed above are continuity-like predicates.

Remark. Obviously, each point of the form $(x, x, f, \varepsilon) \in X \times X \times \mathcal{F}(X, Y) \times \mathbb{R}^*_+$ lies in the set P^{true}. Therefore,

$$\operatorname{diag}(X) \times \mathcal{F}(X, Y) \times \mathbb{R}^*_+ \subset P^{\text{true}},$$

where $\operatorname{diag}(X)$ denotes the diagonal set

$$\{(x, x) | x \in X\} \subset X \times X.$$

The following theorem is a direct generalization of Theorem 1:

Theorem 4 (Malešič–Repovš) *Let P be a continuity-like predicate. Then the following assertions are equivalent:*

(i) There exists a continuous modulus of P-continuity, i.e. a function

$$\hat{\delta} : X \times \mathcal{F}_P(X,Y) \times \mathbb{R}^*_+ \to \mathbb{R}^*_+$$

such that for every

$$(x, x', f, \varepsilon) \in X \times X \times \mathcal{F}_P(X,Y) \times \mathbb{R}^*_+ \to \mathbb{R}^*_+$$

the following implication holds:

$$d(x,x') < \hat{\delta}(x,f,\varepsilon) \Longrightarrow (x,x',f,\varepsilon) \in P^{\text{true}}.$$

(ii) The set $diag(X) \times \mathcal{F}_P(X,Y) \times \mathbb{R}^*_+$ lies in the interior of the set P_0^{true}.

The construction of the modulus

$$\hat{\delta} : X \times \mathcal{F}_P(X,Y) \times \mathbb{R}_+^* \to \mathbb{R}_+^*$$

in the proof of the preceding theorem is exactly as in the proof of Theorem 1:

$$\hat{\delta}(x, f, \varepsilon) = \mathrm{dist}((x, x, f, \varepsilon), P^{\mathrm{false}}).$$

Corollary 3 *There exists a modulus* $\hat{\delta} : X \times \mathcal{C}(X,Y) \times \mathbb{R}_+^* \to \mathbb{R}_+^*$ *of* C*-continuity.*

Proof. We have shown (in the proof of Theorem 1) that the set C^{true} is open if f is continuous. Therefore the condition (ii) is fulfilled and there exists a continuous modulus of C-continuity.

$\square$

Recall that any function defined on a *discrete* metric space X is always continuous, hence every modulus on X is necessarily continuous.

Corollary 4 *Let X be a nondiscrete metric space. Then there are no continuous moduli of USC- (resp. LSC) -continuity on X.*

Proof. Let $x_0 \in X$ be a nonisolated point and define $f : X \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} 1; \text{ if } & x = x_0 \\ 0; \text{ if } & x \neq x_0 \end{cases}$$

Clearly, f is upper semicontinuous, but discontinuous at x_0.

It is easy to see that the point $(x_0, x_0, f, 0.5)$ cannot be an interior point of USC^{true}. By Theorem 4 there are no continuous moduli of USC-continuity. Analogously one can prove that there are no continuous moduli of LSC-continuity.

$\square$

Although a continuous modulus of USC-continuity $\hat{\delta} : X \times \mathcal{C}(X,Y) \times \mathbb{R}_+^* \to \mathbb{R}_+^*$ cannot exist as a function of three variables

$$x \in X, f \in \mathcal{F}_{USC}(X,Y) \ \text{ and } \ \varepsilon \in \mathbb{R}_+^*,$$

there exists a continuous modulus as a function of one variable $\varepsilon \in \mathbb{R}_+^*$:

Corollary 5 *For each upper semicontinuous function* $f : X \to \mathbb{R}$ *and each point* $x \in X$ *there exists a continuous function* $\hat{\delta} : \mathbb{R}_+^* \to \mathbb{R}_+^*$ *such that*

$$(\forall x')d(x', x) < \hat{\delta}(\varepsilon) \Longrightarrow f(x') < f(x) + \varepsilon.$$

Proof. Choose $x \in X$ and $f \in \mathcal{F}_{USC}$ and introduce a function $\tilde{\delta} : \mathbb{R}_+^* \to \mathbb{R}_+^*$ by:

$$\tilde{\delta}(\varepsilon) = \inf\{\delta > 0|$$

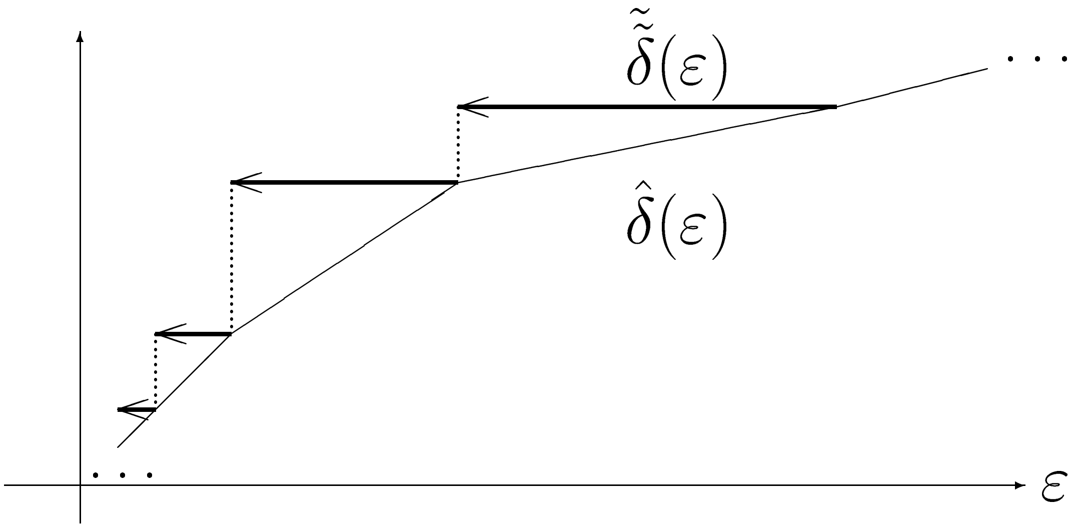
$$(\forall x')d(x', x) < \delta \Longrightarrow f(x') < f(x) + \varepsilon\}.$$

Obviously, $\tilde{\delta}(\varepsilon)$ is a nondecreasing function of ε. There exists a piecewise constant nondecreasing function $\tilde{\tilde{\delta}} : \mathbb{R}_+^* \to \mathbb{R}_+^*$ such that

$$\tilde{\tilde{\delta}}(\varepsilon) \leq \tilde{\delta}(\varepsilon), \text{ for each } \varepsilon \in \mathbb{R}_+^*,$$

e.g. set $\tilde{\tilde{\delta}}(\frac{1}{n}) = \tilde{\delta}(\frac{1}{n})$, for all $n \in \mathbb{N}$.

A continuous piecewise linear function $\hat{\delta}$: $\mathbb{R}_+^* \to \mathbb{R}_+^*$ can be found such that $\hat{\delta}(\varepsilon) \leq \tilde{\tilde{\delta}}(\varepsilon)$, by the construction depicted below:



$\square$

Corollary 6 *Let $\alpha : X \to [0, \infty]$ be a non-lower semicontinuous function and (Y, ρ) a connected metric space of infinite diameter. Then there exist no continuous moduli of C^α-continuity.*

Proof. Let α be nonlower semicontinuous at a point $x_0 \in X$. Since Y is connected and has infinite diameter, it is possible to choose points $y_0, y_1 \in Y$ so that $\rho(y_0, y_1) = \alpha(x_0)$. Define a map $f : X \to Y$ by

$$f(x) = \begin{cases} y_0; \text{ if } \ x = x_0 \\ y_1; \text{ if } \ x \neq x_0 \end{cases}$$

It remains to prove that the point $(x_0, x_0, f, \varepsilon)$ is not an interior point of the set $C^{\alpha,\text{true}}$ for certain values of ε.

$\square$

Up until to now, the modulus of continuity was a function of *three* arguments: the point $x \in X$, the function $f \in \mathcal{F}_P(X, Y)$ and the number $\varepsilon \in \mathbb{R}_+^*$.

Theorem 4 can be generalized to obtain the modulus which depends on *five* arguments: the two additional arguments being the metrics d and ρ. Denote by $\mathcal{M}_X$ the space of all metrics on X, as a subspace of $\mathcal{F}(X \times X, \mathbb{R})$, and $\mathcal{M}_Y$ the space of all metrics on Y, as a subspace of $\mathcal{F}(Y \times Y, \mathbb{R})$, with the induced uniform convergence metric topology. Then a function

$$\hat{\delta} : X \times \mathcal{F}_P \times \mathbb{R}_+^* \times \mathcal{M}_X \times \mathcal{M}_Y \to \mathbb{R}_+^*$$

defined by the relation

$$d(x', x) < \hat{\delta}(x, f, \varepsilon, d, \rho) \Longrightarrow$$
$$\Longrightarrow (x, x', f, \varepsilon, d, \rho) \in P^{\text{true}},$$

for all

$$x, x' \in X, \varepsilon \in \mathbb{R}_+^*, f \in \mathcal{F}_P, d \in \mathcal{M}_X, \rho \in \mathcal{M}_Y,$$

is called a *modulus* of P-continuity.

Theorem 5 (Malešič–Repovš) *There exists a continuous modulus of continuity, i.e. a continuous function*

$\hat{\delta} : X \times \mathbb{R}^*_+ \times \mathcal{C}(X,Y) \times \mathcal{M}_X \times \mathcal{M}_Y \to \mathbb{R}^*_+$

such that

$$d(x', x) < \hat{\delta}(x, \varepsilon, f, d, \rho) \Longrightarrow$$

$$\Longrightarrow \rho(f(x'), f(x) < \varepsilon$$

for each $x', x \in X$, $f \in \mathcal{C}(X,Y)$, $d \in \mathcal{M}_X$ *and* $\rho \in \mathcal{M}_Y$.

Remark. Recently Weihrauch and Zheng found a *computable* modulus of continuity, in the sense of *effective* analysis.