"Ergodic Theory and Set Theory"

Modern algebraic dynamics

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§1 **What and why?**

• "Algebraic dynamics"
$=$ actions of $\mathbf{Z}^d$ by automorphisms of compact groups.

• The higher-rank case shows unexpected new phenomena.

• These systems are essentially the only higher-rank group actions whose dynamics can be understood.

• The theory has thrown up extensive interaction with other fields.

• Actions of higher-rank groups should be studied because of the origins of ergodic theory.

• The algebraic case is hard enough already.

§2 **Dynamics $\leftrightarrow$ commutative algebra.**

Let $\alpha : \mathbf{Z}^d \to \mathsf{Aut}(X)$ be an action by automorphisms of the compact group X. Then $M = \widehat{X}$ (the discrete character group) inherits the structure of an R_d-module, where

$$R_d = \mathbf{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}].$$

For example: $3+u_1+2u_2$ "multiplies" a character $m \in M$ by

$$((3 + u_1 + 2u_2)m)(x) = m\left(3x + \alpha_{(1,0)}x + 2\alpha_{(0,1)}x\right)$$

and so on.

Conversely, if M is an R_d-module, then it determines a $\mathbf{Z}^d$-action α^M on the compact group $X_M = \widehat{M}$.

Fact 1 *Methods from commutative algebra allow many questions about X_M to be reduced to questions about $X_{R_d/P}$.*

The price to pay is that building X_M from $X_{R_d/P}$ is not straightforward.

Recall a prime ideal P is *associated* to M if $P = \mathsf{Ann}_M(m)$ for some $m \in M$; write $\mathsf{Asc}(M)$ for the set of all associated primes.

§3 **Basic results.**

Fix a $\mathbf{Z}^d$-action α by automorphisms of X, Haar measure μ. Then α is (all statements are 'up to null sets'):

• *ergodic* if $\alpha_{\mathbf{n}}(A) = A \Rightarrow \mu(A) = 0$ or 1;

• *mixing on r sets* if for all sets $B_1, \dots, B_r$

$$\lim_{\mathbf{n}_l \in \mathbb{Z}^d \text{ and } \mathbf{n}_l - \mathbf{n}_{l'} \to \infty \text{ for } 1 \le l' < l \le r} \mu\left(\bigcap_{l=1}^{r} T_{-\mathbf{n}_l}(B_l)\right) = \prod_{l=1}^{r} \mu(B_l);$$

• *mixing* if it is mixing on 2 sets;

• *expansive* if there is an open neighbourhood $U \ni 1_X$ with

$$\bigcap_{\mathbf{n} \in \mathbf{Z}^d} \alpha_{\mathbf{n}} U = \{1_X\};$$

• *K* if it has completely positive entropy;

• *Bernoulli* if there is a measurable isomorphism between α and a d–dimensional IID process.

'Entropy' is an important invariant of dynamical systems measuring growth in complexity. For actions of higher-rank groups it is (usually) impossible to compute.

Dynamics of $\alpha^{R_d/P}$	Property of P
expansive	$V_{\mathbb{C}}(P) \cap \mathbf{S}^d = \emptyset$
dense set of periodic points	always
finite set of points with period Λ	$V_{\mathbb{C}}(P) \cap \{\mathbf{c} : \mathbf{c}^{\mathbf{n}} = 1 \forall \mathbf{n} \in \Lambda\} = \emptyset$
ergodic (= has a dense orbit)	$\{\mathbf{u}^{k\mathbf{n}} - 1 : \mathbf{n} \in \mathbf{Z}^d\} \not\subset P \ \forall \ k \geq 1$
mixing	$\mathbf{n} \neq 0 \Rightarrow \mathbf{u}^{\mathbf{n}} - 1 \notin P$
mixing on r sets for all r (= mixing of all orders)*	$P = pR_d$ or $P \cap \mathbf{Z} = \{0\}$ and mixing
positive entropy*	P is principal, and not (generalized) cyclotomic
finite entropy	$P \neq \{0\}$
K	positive entropy
Bernoulli	K
unique maximal measure	$0 <$ entropy $< \infty$

For systems of the form α^M, the dynamical property holds (roughly speaking) if and only if it holds for each $\alpha^{R_d/P}$, $P \in \mathsf{Asc}(M)$.

(In fact the first two statements – the topological ones – also require that the module be Noetherian).

§4 **More on entropy.**

Fact 2 *entropy*($\alpha^{R_d/P}$) $= 0$ *if* P *is non–principal,* ∞ *if* $P = \{0\}$, *and*

$$\int_0^1 \cdots \int_0^1 \log |f(e^{2\pi i s_1}, \ldots, e^{2\pi i s_d})| ds_1 \ldots ds_d$$

if $P = \langle f \rangle$.

This integral is the *Mahler measure*. It is known when it vanishes (higher-dimensional Kronecker lemma).

It is not known what values it takes on.

Example 1 *Let* $P = \langle 1 + u_1 + u_2 \rangle$, $d = 2$. *Then* α *is the shift action of* $\mathbf{Z}^2$ *on*

$$X = \{\mathbf{x} \in \mathbf{T}^{\mathbf{Z}^2} : x_{n,m} + x_{n+1,m} + x_{n,m+1} = 0 \text{ mod } 1 \forall n, m\}.$$

Previous results show that this is a mixing of all orders (indeed, Bernoulli) action with entropy

$$\frac{3\sqrt{3}}{4\pi} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^2},$$

where $\chi(n) = \left(\frac{n}{3}\right)$ *is the Legendre symbol*

$$\chi(n) = \begin{cases} 1 & n \equiv 1 \quad \text{mod } 3, \\ -1 & n \equiv 2 \quad \text{mod } 3, \\ 0 & n \equiv 0 \quad \text{mod } 3. \end{cases}$$

§5 **More on mixing.**

How can something *fail* to be mixing of all orders?

Example 2 *Let* $P = \langle 2, 1 + u_1 + u_2^{-1} \rangle$. *Then* $\alpha^{R_2/P}$ *is the shift action on*

$X = \{\mathbf{x} \in \{0,1\}^{\mathbf{Z}^2} : x_{n,m} + x_{n+1,m} + x_{n,m-1} = 0$ *mod* $2 \forall n, m\}$.

Look at the set A *of elements of* X *that have a* 0 *at* $(0,0)$. *Then* $A \cap \alpha_{(-4,0)} A$ *looks like:*

$$\begin{array}{ccccc}
0 & a & b & c & 0 \\
a & a+b & b+c & c & \\
b & a+c & b & & \\
a+b+c & a+b+c & & & \\
0 & & & &
\end{array}$$

That means:

$$\underbrace{A}_{0@(0,0)} \cap \underbrace{\alpha_{(-4,0)}(A)}_{0@(4,0)} = \underbrace{A}_{0@(0,0)} \cap \underbrace{\alpha_{(-4,0)}(A)}_{0@(4,0)} \cap \underbrace{\alpha_{(0,4)}(A)}_{0@(0,-4)}$$

The same holds for any power of 2, *so* α *is NOT mixing on 3 sets (despite being mixing).*

Mixing of all orders can happen:

• in a dumb way, with $P = pR_d$ and so the system is a full shift (no rules) on $\mathbf{Z}/p\mathbf{Z}^{\mathbf{Z}^d}$, or

• in an interesting way, when the system is mixing and the group X is connected.

Outline:

X connected $\Rightarrow$ can reduce to statements in zero characteristic rings and fields.

Assume a mixing algebraic action on a connected group is NOT mixing on r sets.

Harmonic analysis $\Rightarrow$ EITHER

not mixing on $(r-1)$ sets

OR

find infinitely many non–trivial solutions to an S–unit equation.

Schlickewei $\Rightarrow$ mixing of all orders by induction.

§6 **Order of mixing.**

Example 2 above is typical: if $P \ni p$, some prime, then any polynomial $f \in P$ forces *failure* to mix on dilates by p^n of the support of f for all $n \geq 1$.

The *order of mixing* is the largest r for which the system is mixing on r sets.

Thus, if $P = \langle p, f \rangle$ then

order of mixing $\leq$ number of terms in f-1.

Much is known, but the central question (when is this an equality?) is open.

Example 3 *When $p = 2$, the polynomial*

$$f = 1 + u_1 + u_1^3 + u_2 + u_1 u_2 + u_1 u_2^2 + u_2^3$$

gives a system whose order of mixing is 2.

§7 **Classification and 'abelian rigidity'.**

Measurable: when is there an isomorphism of measure spaces $X_M \to X_N$ that intertwines the actions?

Example 4 *The systems corresponding to*

$$R_2/\langle 1+u_1+u_2\rangle \text{ and } R_2/\langle 1+u_1+u_2^{-1}\rangle$$

are isomorphic.

The systems corresponding to

$$R_2/\langle 2, 1+u_1+u_2\rangle \text{ and } R_2/\langle 2, 1+u_1+u_2^{-1}\rangle$$

are NOT isomorphic.

Topological: when is there a homeomorphism $X_M \to X_N$ that intertwines the actions?

When the groups are connected, such a homeomorphism exists if and only if there is an algebraic isomorphism, which reduces the problem to algebra.

In the disconnected case less is known.

Notice that each *measurable* isomorphism class splits into uncountable many *topological* equivalence classes.

Abelian rigidity: it is difficult to commute with many things.

Non-abelian rigidity involves results like that of Zimmer: orbit equivalence (even weaker than measurable isomorphism) for actions of connected simple Lie groups of rank > 1 forces the groups to be isomorphic and the actions to be isomorphic.

The 'abelian rigidity' seen here only occurs because of the algebraic nature of the examples.

Example 5 *Fix* $P = \langle 2, 1 + u_1 + u_2 \rangle$, *and let* $\phi : X_{R_2/P} \to X_{R_2/P}$ *be a measure preserving Borel map that commutes with* α. *Then* ϕ *coincides a.e. with a continuous, surjective homomorphism* $\psi : X_{R_2/P} \to X_{R_2/P}$.

A corollary is that the only invertible measure preserving maps that commute with the action must be a.e. members of the action themselves.

Fact 3 *There are many measurable maps that commute with any single* $\alpha_{\mathbf{n}}$.

The current state of knowledge is this. If α_1 and α_2 are algebraic dynamical systems with $d > 1$ that are mixing, expansive and irreducible, then any *measurable* isomorphism between them is *affine*.

§8 **Invariant measures.**

Furstenberg showed that a closed infinite subset of the circle $\mathbf{T}$ invariant under $\times 2$ and $\times 3$ mod 1 must be all of $\mathbf{T}$.

What non-atomic measures in an algebraic dynamical system are invariant under several group automorphisms that generate a mixing action?

The basic conjecture is that:

invariance $\Rightarrow$ *measure is Haar.*

Known results on connected groups mostly say:

invariance $+$ *entropy condition* $\Rightarrow$ *measure is Haar.*

With one interesting exception:

invariance on certain zero-dimensional examples $+$ *being sufficiently mixing* $\Rightarrow$ *measure is Haar.*

§9 **Open problems.**

•1 Classify the systems up to measurable isomorphism (known when $d = 1$).

•2 What kinds of equivalence relations can be embedded in ... that classification? ... the different kinds of mixing breakdown?

•3 Determine the exact order of mixing of, say, the system with $P = \langle 2, 1 + u_1 + u_2 + u_1^2 \rangle$.

•4 Find the set of possible entropies (= 'Lehmer's problem').

•5 What are the invariant measures?

•6 Find a more reasonable proof of mixing of all orders.

•7 Construct and interpret symbolic dynamics for these systems.

•8 Understand directional phenomena for these systems. This involves 'directional commutative algebra'.

•9 If μ is a probability measure on the circle that makes $\times 2$, $\times 3$ into a mixing of all orders system, is $\mu =$ Lebesgue?

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