The Length of Infinite Time Turing Machine Computations

P. D. Welch

Graduate School of Science & Technology,
Kobe University,
Rokko-dai, Nada-ku
Kobe 657, Japan.
welch@pascal.seg.kobe-u.ac.jp

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§1 Machine theoretic definitions and results.

§2 Getting Clockables = Writables.

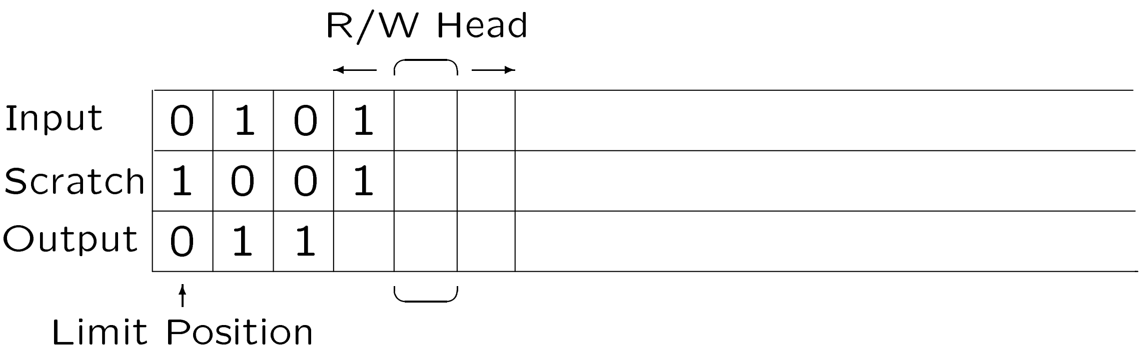
§3 The set-theoretic context.

Reference:
"Infinite Time Turing Machines" J. Hamkins & A. Lewis, J. of Symbolic Logic (to appear).

§1 The H-K machines

Hardware: just as for ordinary Turing Machines, but for convenience, we consider the infinite tape as divided into three infinite tapes, for input, scratch computations and an output tape.

One tape position is designated in advance, as the "Limit Position".



Software: just as for ordinary Turing Machines, but with the addition of an extra "limit state" q_L.

Let $\langle P_n \,|\, n < \omega \rangle$ enumerate all programs. Let the cells be enumerated as $\langle C_i \,|\, i < \omega \rangle$ with contents at time ν in the computation of program n denoted by $C_i^n(\nu) \in \{0, 1\}$.

Action: *At times* $\nu = \beta + 1$: obey the usual T.M. rules of the program P_n.

At limit times ν: R/W Head returns to the Limit Position; Cell values $C_i(\nu) = Lim_{\overline{\tau} \to \nu} \sup_{\tau \geq \overline{\tau}} C_i(\tau)$.

• $\varphi_n(x)(x \in 2^\omega)$ be the result of running P_n on input string x.

• $\forall n, x \exists \alpha < \omega_1$ so that the behaviour of $\varphi_n(x)$ has "settled" by stage α: either it has entered a permanent looping cycle or it has halted: $\varphi_n(x){\downarrow}y$ for some $y \in 2^\omega$.

• Decidable properties: Arithmetic; "$x \in WO$"; (thus) any Π^1_1 predicate.

Def $\xi \in \mathcal{W} \longleftrightarrow \exists p \in \omega \phi_p(0){\downarrow}x$ $\quad$ ($\mathcal{W}$ = "Writables")

Note: "$\varphi_p(x){\downarrow}y$ (in α steps)", "$\varphi_p(x){\uparrow}$" are both Δ^1_2 properties (in a code for α).

Def $\lambda =_{df} \sup\{||x|| : x \in WO \cap \mathcal{W}\}; \gamma =_{df} \sup\{\text{Clock}\}$.
Clock $=_{df} \{\alpha \mid \exists p \in \omega\ \phi_p(0)$ in exactly α steps $\}$.

Fact (Hamkins-Lewis) $\lambda \leq \gamma$.

Q.1 Is $\lambda = \gamma$?

Def $H_\lambda = \{y \in HC \mid \exists \overline{y} \in \mathcal{W} \wedge \overline{y}$ codes $y\}$.

Q.2 Is $H_\lambda = L_\lambda$?

Fact (Hamkins-Lewis) H_λ is a transitive admissible set, closed under hyperjump (and more).

Oracles: For $y \in 2^\omega \quad \varphi^y_p(x) \simeq \varphi_{p'}(\langle x, y\rangle)$
For $A \subseteq 2^\omega \quad \varphi^A_p(x)$. Ask Yes/No questions "Is $z \in A$?"

There is an associated *reducibility relation* associated to reals or sets of reals: $x \leq_\infty y, A \leq_\infty B$:

$$x \leq_\infty y \longleftrightarrow \exists n \varphi_n(x){\downarrow} y$$

$$A \leq_\infty B \longleftrightarrow \exists n \forall x \varphi_n^B(x){\downarrow}1 \text{ if } x \in A, \ {\downarrow}0 \text{ if } x \notin A$$

Jumps: $O^\triangledown =_{df} \{n \mid \varphi_n(0){\downarrow}\}$ ("Weak Jump").

$O^\blacktriangledown =_{df} \{(n,x) \in \omega \times 2^\omega \mid \varphi_n(x){\downarrow}\}$ ("Strong Jump").

Def $\boldsymbol{\Gamma} =_{df} \{A \subseteq 2^\omega \mid A$ is semi-decidable from a real $\} =$
$= \{A \subseteq 2^\omega \mid \exists r \in 2^\omega \exists n \forall x \quad x \in A \longleftrightarrow \varphi_n^r(x){\downarrow}1\}$.
$\boldsymbol{\Delta} = \boldsymbol{\Gamma} \cap \neg \boldsymbol{\Gamma}$.

Fact (Hamkins-Lewis) $\boldsymbol{\Delta} \subseteq \boldsymbol{\Delta}^1_2$ is a σ-algebra, closed under Suslin's operation $\mathcal{A}$. $\boldsymbol{\Gamma}$ forms a Spector pointclass with norm $\psi : A \longrightarrow On$ on a semi-decidable set $A = \{x \in 2^\omega \mid \varphi_p(x){\downarrow}\}$:

$$\psi(x) \simeq \alpha(\varphi_p(x){\downarrow} \text{ in } \alpha \text{ steps. })$$

Q.3 What are the decidable sets of integers? What are the semi-decidable sets of reals?

§2 **Clockables are Writable**

Theorem 1 $\lambda = \gamma$.

This is a corollary of the Main Proposition below about stabilization times of cells.

Def (H-L) (i) *The* eventually writable *reals:* $\mathcal{EW} =_{df} \{x \mid \exists p \exists \mu \forall \nu \geq \mu \varphi_p(0)$ *has* x *on its output tape at time* $\nu\}$.
(ii) The accidentally writable *reals:* $\mathcal{AW} =_{df} \{x \mid \exists p \exists \mu \varphi_p(0)$ *has* x *on its output tape at time* $\mu\}$;
(iii) $\zeta =_{df} \sup\{||x|| : x \in \mathcal{EW} \cap WO\}$;
(iv) $\Sigma =_{df} \sup\{||x|| : x \in \mathcal{AW} \cap WO\}$.

Then $\mathcal{W} \subseteq \mathcal{EW} \subseteq \mathcal{AW}$ and one can show that the inclusions are strict; indeed H-L show $\lambda < \zeta < \Sigma$.

Fix an arbitrary $p \in \omega$ and consider the computation $\varphi_p(0)$ and the cells $C_i = C_i^p$ of that machine's tape.

Def *For any time* ν *let*

$$\delta_i(\nu) = \sup\{\nu' < \nu \mid \nu' = 0 \vee C_i(\nu') \neq C_i(\nu'+1)\} \leq \nu.$$

For Lim(ν) we say C_i *has* (locally) stabilized at stage ν, *if* $\delta_i(\nu) < \nu$.

Main Proposition $\forall i\ \delta_i(\Sigma) < \Sigma \longrightarrow \delta_i(\Sigma) < \zeta$.

Proof: Consider $\mathcal{U}$ the universal machine that simulates, by cutting up its scratch tape into infinitely many slices, all the computations $\phi_q(0)$ for each q simultaneously.

Def *Let* $\sigma(\nu) =_{df}$
$\Sigma\{||x|| : x \in WO \wedge x$ *is on one of* $\mathcal{U}$*'s tapes at stage* $\nu\}$.

(1) As $\nu \to \infty$ and as $\sigma(\nu) \in \mathcal{AW}$, $\sigma(\nu)$ becomes unbounded in Σ.
(2) Once ν is sufficiently large, all the tapes of $\mathcal{U}$ that are destined to be *eventually written* to, will have received those eventual values; hence at some earliest moment $\overline{\nu}$ say, $\sigma(\overline{\nu}) \geq \zeta$, and this will hold at all later stages of time after $\overline{\nu}$.

We describe now a machine $\mathcal{M} = \mathcal{M}(i,p)$, whose scratch tape is divided into three infinite parts S_0, S_1, S_2. $\mathcal{M}$ is simulating the universal program $\mathcal{U}$ on S_0 (by further slicing up of S_0). Let T denote $\mathcal{M}$'s output tape. $\mathcal{M}$ is to behave as follows at time ν.
(i) $\mathcal{M}$ computes a current value for "$\sigma(\nu)$" by looking at the cells of its "$\mathcal{U}$";
(ii) on its scratch tape S_1, it writes a code for the whole course-of-computation of the first $\sigma(\nu)$ steps of $\phi_p(0)$. Let $\delta = \delta_i(\sigma(\nu))$ the current stabilization value for cell C_i.
(iii) Then $\mathcal{M}$ inspects T: the contents of T is a sequence $x \in 2^\omega$, it tests (using S_2) if $x \in WO$. If (a) x was not a code for a wellorder, or if (b) $||x|| < \delta$, then x on T is overwritten by (a code for) δ (which $\mathcal{M}$ has available); otherwise T is left untouched.
(iv) $\mathcal{M}$ runs another step in each of $\mathcal{U}$'s computations, and then returns to stage (i) now at time $\nu + 1$.

Claim If $\delta_i(\Sigma) < \Sigma$ then (a) the content of T is eventually constant, and codes an ordinal $\tau < \zeta$; (b) $\delta_i(\Sigma) \leq \tau$.

Proof: Let $\tilde{\delta} =_{df} \delta_i(\Sigma)$ and suppose $\tilde{\delta} < \Sigma$. At various stages ν, the current $\sigma(\nu)$ is calculated. When $\mathcal{M}$ inspects T at (iii) above, it always leaves a code z on T with $||z|| \geq \delta$ each time.

Eventually (by (1) above) $\sigma(\nu))$ becomes sufficiently large below Σ that it is greater than the true $\tilde{\delta}$. At *this* time an ordinal $\geq \tilde{\delta}$ ends up written on T. But note the content of T remains fixed hereafter: there will not be any reason to overwrite it, irrespective of the fact that the value of $\sigma(\nu)$ fluctuates. This is our τ. Hence $\tau \in \mathcal{EW}$ as witnessed by the machine $\mathcal{M}$. Thus $\tilde{\delta} \leq \tau < \zeta$.

Q.E.D.Claim & Proposition

Cor *Theorem 1 holds.*
Proof: We have just seen that for any $i < \omega$ if C_i is stabilized at Σ, then the point of stabilization is below ζ. Hence a), the "snapshot" of the state of the computation (meaning the distribution of 0's and 1's in the cells C_i^p, the state of the machine, etc.) is exactly the same at time ζ and Σ; *and moreover* b), for no $i < \omega$ for which $C_i(\zeta) = 0$, is $C_i(\xi) = 1$ for any $\zeta < \xi < \Sigma$. The conclusion to be drawn from b) is that either $\varphi_p(0)\downarrow$, and has halted at time $\mu < \zeta$, (so μ is "clockable", i.e. $\mu < \gamma$), or the snapshots at ζ and Σ are from a *permanently* looping cycle.

Claim If $\varphi_p(0)\downarrow$then $\mu < \lambda$.
Proof: As $\mu < \zeta$ let $q \in \omega$ be a program number so that $\phi_q(0)$ eventually writes μ on its output tape. Informally we let P_r be the program that (i) runs $\varphi_q(0)$; (ii) keeps checking to see if O.T. of $\varphi_q(0)$ contains an ordinal δ long enough to support a halting computation of $\varphi_p(0)$. When (ii) occurs P_r is to halt with output that ordinal δ. Hence $\mu \leq \delta \in \mathcal{W}$. So $\mu < \lambda$. **Q.E.D.Claim**

Hence every "clockable" ordinal is "writable". **Q.E.D.Cor**

By similar arguments:

Cor $H_\lambda = L_\lambda; L_\zeta = \{y \,|\, y$ *is coded by some* $\overline{y} \in \mathcal{EW}\}$.

Cor *The decidable sets of integers are those* $x \in 2^\omega \cap L_\lambda$.

§3 The Set-Theoretic Context

Theorem 2 *(ζ, Σ) is the lexicographically least pair with ζ Σ_2-extendible to Σ;* i.e. $L_\zeta \prec_{\Sigma_2} L_\Sigma$.

Proof: Sketch of $L_\zeta \prec_{\Sigma_2} L_\Sigma$: suppose $L_\Sigma \models \exists x \forall y \psi(x, y, u)$ with $u \in L_\zeta$, $\psi \in \Sigma_0$. Let x_0 be the L_Σ-least witness. Let ρ_0 be its L-rank. As $u \in L_\zeta, u \in \mathcal{EW}$. Fix a computation $\varphi_q(0)$ that eventually has (a code for) u on its output tape. Suppose that at time ν the contents of this tape is $u(\nu)$.

We describe a machine similar to the above: let $\mathcal{M}$ run the universal machine $\mathcal{U}$ and compute $\sigma(\nu)$ as above. We let $x(\nu)$ denote the contents of $\mathcal{M}$'s O.T. at time ν. Given $\sigma(\nu)$, $\mathcal{M}$ computes a code for $L_{\sigma(\nu)}$ and checks if $L_{\sigma(\nu)} \models \forall y \psi(x(\nu), y, u(\nu))$ (and is the L-least such); If so, or if $x(\nu)$ is not a code for any set in $L_{\sigma(\nu)}$ then $\mathcal{M}$ leaves the O.T. unchanged and sets $x(\nu + 1) = x(\nu)$. Otherwise it writes a code for the current $L_{\sigma(\nu)}$-least witnessing x to its own O.T. as $x(\nu + 1)$. As $\sigma(\nu)$ is unbounded in Σ, eventually both $u = u(\nu)$ and $\sigma(\nu) > \rho_0$. At this point a code for x_0 is written to the O.T. This will never be overwritten (even were $\sigma(\nu)$ to drop below ρ_0). Hence $x_0 \in \mathcal{EW}$, and so $x_0 \in L_\zeta$ by the results above.

Leastness: check that if $L_{\overline{\zeta}} \prec_{\Sigma_2} L_{\overline{\Sigma}}$ then $\overline{\zeta}$ and $\overline{\Sigma}$ have "permanently repeating snapshots" of any computation.

Q.E.D.

Theorem 3 *(i)* λ *is the least* λ' *with* $L_{\lambda'} \prec_{\Sigma_1} L_\zeta$ *;*
(ii) $O^\triangledown \equiv_1 A^1_\lambda$ *(the* Σ_1*-mastercode for* L_λ*).*

Cor $x \leq_\infty y \longleftrightarrow x \in L_{\lambda^y}[y]$*; the assignment* $x \to \lambda^x$ *satisfies a Spector Criterion, thus:*

$$x \leq_\infty y \longrightarrow (x^\triangledown \leq_\infty y \longleftrightarrow \lambda^x < \lambda^y).$$

• This lets us develop a degree theory on reals that is somewhere intermediate between Δ^1_1-degrees (or *hyperdegrees*) and Δ^1_2-degrees.

• We may characterise the jump hierarchy in terms of ζ-stable ordinals; further we may characterise the reals in the iterates of the (weak) jump operator $0^{\triangledown\alpha}$ for $\alpha < \zeta$; and the reals $0^{\triangledown\alpha}$ themselves as certain mastercodes.

• In particular we may answer various questions concerning minimal ∞-degrees.

The strength of the machine architecture

Kidder and Hamkins have defined the operation Γ_{HL}:

$$\Gamma_{HL}(\langle\langle C_i(\tau) \,|\, i<\omega\rangle \,|\, \tau \leq \nu\rangle) = \langle C_i(\nu+1) \,|\, i<\omega\rangle$$

as being determined by the underlying Turing machine program at successor stages, whilst $\langle C_i(\nu) \,|\, i<\omega\rangle =$

$$\Gamma_{HL}(\langle\langle C_i(\tau) \,|\, i<\omega\rangle \,|\, \tau < \nu\rangle) = \langle \lim_{\tau\to\nu} \sup_{\tau<\tau'<\nu} C_i(\tau') \,|\, i<\omega\rangle$$

at limit stages ν. More generally, let us call any function $\Gamma : {}^{<On}({}^{\omega}2) \Rightarrow ({}^{\omega}2)$ defined on sequences of reals "appropriate", and let us call a "Γ-machine", one that uses the operator Γ at all limit stages. Let $\mathcal{W}_\Gamma$, $\mathcal{EW}_\Gamma \ldots$ denote the classes of reals arising when we use such a machine.

Def *A Γ-machine is Σ_n-complete, if for any appropriate $\overline{\Gamma}$, provably total in Σ_n-KP, then $\mathcal{EW}_\Gamma \supseteq \mathcal{EW}_{\overline{\Gamma}}$.*

Theorem 4 *(i) Γ_{HL} is Σ_2-complete.*

(ii) For appropriate Σ_2-KP definable Γ, $\mathcal{W} \supseteq \mathcal{W}_\Gamma$.

(iii) However there is a provably total Γ^* in Σ_2-KP $+\Sigma_2$-Separation, so that $\mathcal{W}_{\Gamma^*} \supset \mathcal{EW}$.

Higher-Type Reducibility

It follows from the above that the sets of reals semi-decidable in $x \in 2^\omega$ are those A expressible as follows: let $\psi_0(v_0, v_1)$ be a $\mathbf{\Sigma}_1$ formula, then let

$$y \in A \Longleftrightarrow L_{\lambda^{\langle x,y\rangle}}[x, y] \models \psi_0(x, y).$$

A is *semi-decidable in* B *(*$A, B \subseteq 2^\omega$*) and* $x \in 2^\omega$ can be seen to be given similarly as:

$$y \in A \Longleftrightarrow L_{\lambda^{\langle x,y,B\rangle}}[x, y, B] \models \psi_0(x, y, B).$$

Let $A \leq_{r,\infty} B$ denote A is infinite time computable from a real and B.

Post's problem: are there
1) $A \subseteq 2^\omega$, $A \in \mathbf{\Gamma}, A$ not decidable but $0^\blacktriangledown \not\leq_{r,\infty} A$?
2) $A, B \in \mathbf{\Gamma}$ $A \not\leq_{r,\infty} B \wedge B \not\leq_{r,\infty} A$?

• In L, the constructible universe, there are A, B as in 2).

• Det(Boolean($\mathbf{\Gamma}$)) $\Rightarrow$ any $A \in \mathbf{\Gamma} \setminus \mathbf{\Gamma} \cap \neg\mathbf{\Gamma}$ satisfies $0^\blacktriangledown \equiv_{r,\infty} A$.

Open Questions

Open Question 1
Does a negative answer to 1) imply $Det(\mathbf{\Pi}^1_1)$? Or $Det(\mathbf{\Gamma})$?

Open Question 2
Does *Wadge*-$Det(\mathbf{\Gamma}) \Rightarrow Det(\mathbf{\Gamma})$?

Open Question 3
What is the strength of $Det(\mathbf{\Gamma})$?

We do know that

- $Det(\mathbf{\Gamma}) \Rightarrow \exists\#$ for an inner model of a strong cardinal.
- In fact, an example of a decidable predicate is:

  "$\langle x, y\rangle \in 2^\omega$ codes countable premice $M, N <_* O^\P$ and $M \leq_* N$ in the (pre)mouse ordering."