Solutions to Problem Sheet 2

1. Let $u = \cos 2x$ and $v = x^2 - 1$. Then by Leibnitz's Rule,

$$\begin{aligned}
\frac{d^6}{dx^6}[(x^2-1)\cos 2x] &= \frac{d^6}{dx^6}(\cos 2x)(x^2-1) \\
&+ \binom{6}{1}\frac{d^5}{dx^5}(\cos 2x)(2x) \\
&+ \binom{6}{2}\frac{d^4}{dx^4}(\cos 2x)2 + 0.
\end{aligned}$$

But

$$\begin{aligned}
\frac{d^6}{dx^6}(\cos 2x) &= -2^6\cos 2x, \\
\frac{d^5}{dx^5}(\cos 2x) &= -2^5\sin 2x, \\
\frac{d^4}{dx^4}(\cos 2x) &= 2^4\cos 2x,
\end{aligned}$$

hence

$$\begin{aligned}
\frac{d^6}{dx^6}[(x^2-1)\cos 2x] &= 1\cdot(-2^6\cos 2x)(x^2-1) \\
&+ 6(-2^5\sin 2x)\cdot(2x) \\
&- \frac{6\cdot 5}{\cancel{2}}(2^4\cos 2x)\cdot\cancel{2}
\end{aligned}$$

which eventually reduces to

$$\begin{aligned}
&= 16[(x^2-1)\,(-4\cos 2x) - 24x\sin 2x + 30\cos 2x] \\
&= 16[(34-4x^2)\cos 2x - 24x\sin 2x].
\end{aligned}$$

2. (a) First, we evaluate

$$\frac{dy}{dt} = 1 - \cos t, \qquad \frac{dx}{dt} = \sin t.$$

Then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sin t}{1 - \cos t},$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) / \frac{dx}{dt},$$

so

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dt}\left(\frac{\sin t}{1 - \cos t}\right) / (1 - \cos t) \\ &= \frac{(1 - \cos t)\cos t - \sin^2 t}{(1 - \cos t)^3} \\ &= \frac{\cos t - \cos^2 t - \sin^2 t}{(1 - \cos t)^3},\end{aligned}$$

where the quotient rule has been used to find $\frac{d}{dt}\left(\frac{dy}{dx}\right)$.

But $\sin^2 t + \cos^2 t \equiv 1$, therefore:

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{\cos t - 1}{(1 - \cos t)^3} \\ &= \frac{-((1 - \cos t))}{(1 - \cos t)^3} \\ &= -\frac{1}{(1 - \cos t)^2}.\end{aligned}$$

(b) If we take a closer look at our answer for part (a), i.e.

$$\frac{d^2y}{dx^2} = -\frac{1}{(1-\cos t)^2},$$

this must always be negative (as the square of something must always be positive). Therefore the curve is concave.

(c) Since we already deduced that $\frac{d^2y}{dx^2} < 0$ regardless of the value of t, we can immediately declare that any stationary point is guaranteed to be a *maximum*. Now

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sin t}{1-\cos t} = 0$$

when

$$\sin t = 0, \quad \text{For}\, t = 0, \pi, 2\pi, ldots.$$

Note that if you put $t = 0$ or $t = 2\pi$ (or any other even multiple of π), you end up with a division by zero for $\frac{dy}{dx}$, so it does not give a stationary point! We have more luck with $t = \pi$; this gives $x = \pi$, $y = 2$. So we have a maximum at $(\pi, 2)$.
Remark: If you check $t = 3\pi$, 5π, $\ldots$, then similarly these are also maxima with $y = 2$.

3. Using the Quotient Rule with $u = x + 2$, $v = x^2 + 4x + 5$,

we compute

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x^2+4x+5)(1)-(x+2)(2x+4)}{(x^2+4x+5)^2} \\ &= \frac{(x^2+4x+5)-(2x^2+8x+8)}{(x^2+4x+5)^2} \\ &= \frac{-x^2-4x-3}{(x^2+4x+5)^2},\end{aligned}$$

which equals zero only when

$$x^2+4x+3=(x+3)(x+1)=0,$$

which occurs for $x=-3,-1$ (where $y=-0.5,0.5$ respectively). Therefore these are the two stationary points for the function, and the easiest way to classify them is by applying the sign test on $\frac{dy}{dx}$, see Table 1.

x	$x<-3$	$-3<x<-1$	$x>-1$
$\frac{dy}{dx}$	$-$	$+$	$-$
Slope	\	/	\

Table 1: The sign test on $\frac{dy}{dx}$ is used here to classify the stationary points.

From this test we deduce that there is a minimum at $x=-3$, while $x=-1$ is a maximum.

Meanwhile, observe that the denominator of the function is $x^2+4x+5=(x+2)^2+1$, which has no real roots.

Hence there are no vertical asymptotes. However,

$$\text{as} \qquad x \to \infty, \qquad y \to 0^+,$$
$$\text{as} \qquad x \to -\infty, \qquad y \to 0^-,$$

therefore there is a horizontal asymptote, and its equation is $y = 0$, as seen in Figure 1.

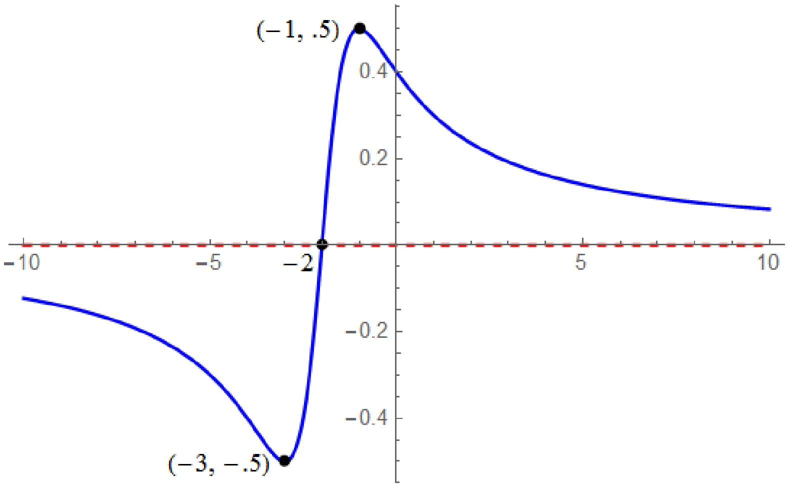

Figure 1: A plot of the curve for Question 3. Observe that the positions of the maximum and minimum have been made clear. If you didn't draw the horizontal asymptote, don't panic! I will not remove any marks, since it clashes with the $x-$axis.

4. (a) First, we require the y coordinate of the point, which is found by letting $x = 1$ in the implicit equation:

$$y^2(2 - x) = x^3 \qquad \Rightarrow \qquad y^2 = 1.$$

This gives $y = -1, 1$. Since we are given that y is positive, we choose $y = 1$. Thus the coordinates of the point are $(1, 1)$. Next, we need the slope of the tangent to $(1, 1)$. To obtain this, start by implicitly differentiating $y^2(2-x) = x^3$:

$$2y(2-x)\frac{dy}{dx} - y^2 = 3x^2,$$

which rearranges to

$$\frac{dy}{dx} = \frac{3x^2 + y^2}{2y(2-x)}.$$

For $x = 1$, $y = 1$, this gives

$$\frac{dy}{dx} = \frac{4}{2} = 2,$$

so $m = 2$ is the slope of the tangent.
To find the equation of the tangent, let $x_1 = 1$ and $y_1 = 1$ in

$$y - y_1 = m(x - x_1),$$

which gives

$$y - 1 = 2(x-1) \quad \Rightarrow \quad y = 2x - 1.$$

Meanwhile, the slope of the normal is

$$-\frac{1}{\text{Slope of tangent}} = -\frac{1}{2},$$

hence the normal satisfies

$$y - 1 = -\frac{1}{2}(x-1),$$

which yields

$$y = \frac{1}{2}(3 - x).$$

(b) Start by implicitly differentiating both sides. This gives:

$$2x + 2y + 2x\frac{dy}{dx} - 6y\frac{dy}{dx} = 0.$$

We can substitute $x = 1$ and $y = 1$ into the above equation straightaway. The result is:

$$2 + 2 + 2\frac{dy}{dx} - 6\frac{dy}{dx} = 0 \quad \Rightarrow \quad 4 - 4\frac{dy}{dx} = 0,$$

which shows that

$$\frac{dy}{dx} = 1,$$

and thus the slope of the normal is $-\frac{1}{1} = -1$.

Now recall that the normal satisfies the equation

$$y - y_1 = m(x - x_1),$$

where $m = -1$, $x_1 = 1$ and $y_1 = 1$. So

$$y - 1 = -(x - 1), \quad \Rightarrow \quad y = 2 - x$$

is the equation of the normal.

To find the points where the normal intersects the curve, we need to solve the simultaneous equations

$$x^2 + 2xy - 3y^2 = 0, \quad \text{and} \quad y = 2 - x.$$

Substituting $y = 2 - x$ into the other equation gives

$$x^2 + 2x(2-x) - 3(2-x)^2 = 0$$

$$\begin{aligned}
&\Rightarrow -4x^2 + 16x - 12 = 0 \\
&\Rightarrow 4x^2 - 16x + 12 = 0 \\
&\Rightarrow x^2 - 4x + 3 = 0 \\
&\Rightarrow (x-1)(x-3) = 0 \\
&\Rightarrow x = 1, 3.
\end{aligned}$$

We already know one point with $x = 1$, i.e. $(1, 1)$, but we want the other point! Therefore we should choose $x = 3$. Putting $x = 3$ into the equation for the normal gives

$$y = 2 - 3 = -1.$$

Hence the other point where the normal intersects the curve is $(3, -1)$.

5. (a) If we differentiate once, the Chain Rule tells us that

$$\frac{ds}{dt} = -2\pi bA\sin(2\pi bt), \tag{1}$$

which is the velocity. If we then differentiate a second time, we have the following:

$$\frac{d^2s}{dt^2} = -(2\pi b)^2 A\cos(2\pi bt), \tag{2}$$

This is the acceleration of the piston. Finally, we will differentiate once more to get the third derivative...

$$\frac{d^3s}{dt^3} = (2\pi b)^3 A\sin(2\pi bt), \tag{3}$$

. . . and this is the jerk of the piston.

(b) Looking at the equations, we see that the velocity, acceleration and jerk have factors of b, b^2 and b^3 respectively. So when the frequency b is doubled. . .

- The velocity doubles.
- The acceleration is multiplied by a factor of $2^2 = 4$.
- The jerk is multiplied by a factor of $2^2 = 8$ (which is massive!)