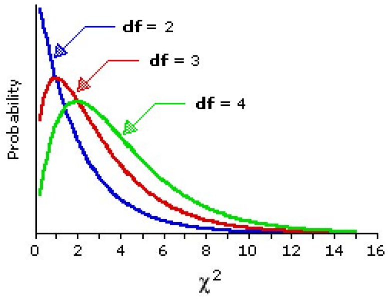


Figure 3: Chi square distributions with different degrees of freedom

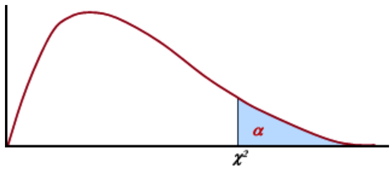


Figure 4: χ^2 distribution with ν degrees of freedom

2 Lecture 2

2.1 The chi square distribution

In particular, when $\alpha = \nu/2$ and $\beta = 2$, we have the **chi square distribution** (χ^2) with ν degrees of freedom.

We use $\chi^2(\nu)$ to denote a random variable having a chi square distribution with ν degrees of freedom, Figure 3. Note that random variables $\chi^2(\nu)$ can only take on non-negative values. So their probability density function is **not** symmetric.

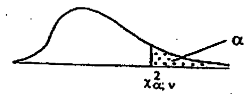
Example 7 *Show that the probability density function of a chi square distribution with ν degrees of freedom is*

$$f(x) = \begin{cases} \frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{\frac{\nu}{2}-1} e^{-x/2}, & x > 0 \\ 0 & x \leq 0, \end{cases} \tag{23}$$

where the gamma function Γ is given by Eq. (18).

Solution. Eq. (23) is an immediate consequence of Eq. (17).

Table of the Chi-square Distribution



α =	0.995	0.99	0.98	0.975	0.95	0.90	0.80	0.20	0.10	0.05	0.025	0.02	0.01	0.005	0.001	= α
v = 1	0.0000393	0.000157	0.000628	0.000982	0.00393	0.0158	0.0642	1.642	2.706	3.841	5.024	5.412	6.635	7.879	10.827	v = 1
2	0.0100	0.0201	0.0404	0.0506	0.103	0.211	0.446	3.219	4.605	5.991	7.378	7.824	9.210	10.597	13.815	2
3	0.0717	0.115	0.185	0.216	0.352	0.584	1.005	4.642	6.251	7.815	9.348	9.837	11.345	12.838	16.268	3
4	0.207	0.297	0.429	0.484	0.711	1.064	1.649	5.989	7.779	9.488	11.143	11.668	13.277	14.860	18.465	4
5	0.412	0.554	0.752	0.831	1.145	1.610	2.343	7.289	9.236	11.070	12.832	13.388	15.086	16.750	20.517	5
6	0.676	0.872	1.134	1.237	1.635	2.204	3.070	8.558	10.645	12.592	14.449	15.033	16.812	18.548	22.457	6
7	0.989	1.239	1.564	1.690	2.167	2.833	3.822	9.803	12.017	14.067	16.013	16.622	18.475	20.278	24.322	7
8	1.344	1.646	2.032	2.180	2.733	3.490	4.594	11.030	13.362	15.507	17.535	18.168	20.090	21.955	26.125	8
9	1.735	2.088	2.532	2.700	3.325	4.168	5.380	12.242	14.684	16.919	19.023	19.679	21.666	23.589	27.877	9
10	2.156	2.558	3.059	3.247	3.940	4.865	6.179	13.442	15.987	18.307	20.483	21.161	23.209	25.188	29.588	10
11	2.603	3.053	3.609	3.816	4.575	5.578	6.989	14.631	17.275	19.675	21.920	22.618	24.725	26.757	31.264	11
12	3.074	3.571	4.178	4.404	5.226	6.304	7.807	15.812	18.549	21.026	23.337	24.054	26.217	28.300	32.909	12
13	3.565	4.107	4.765	5.009	5.892	7.042	8.634	16.985	19.812	22.362	24.736	25.472	27.688	29.819	34.528	13
14	4.075	4.660	5.368	5.629	6.571	7.790	9.467	18.151	21.064	23.685	26.119	26.873	29.141	31.319	36.123	14
15	4.601	5.229	5.985	6.262	7.261	8.547	10.307	19.311	22.307	24.996	27.488	28.259	30.578	32.801	37.697	15
16	5.142	5.812	6.614	6.908	7.962	9.312	11.152	20.465	23.542	26.296	28.845	29.633	32.000	34.267	39.252	16
17	5.697	6.408	7.255	7.564	8.672	10.085	12.002	21.615	24.769	27.587	30.191	30.995	33.409	35.718	40.790	17
18	6.265	7.015	7.906	8.231	9.390	10.865	12.857	22.760	25.989	28.869	31.526	32.346	34.805	37.156	42.312	18
19	6.844	7.633	8.567	8.907	10.117	11.651	13.716	23.900	27.204	30.144	32.852	33.687	36.191	38.582	43.820	19
20	7.434	8.260	9.237	9.591	10.851	12.443	14.578	25.038	28.412	31.410	34.170	35.020	37.566	39.997	45.315	20
21	8.034	8.897	9.915	10.283	11.591	13.240	15.445	26.171	29.615	32.671	35.479	36.343	38.932	41.401	46.797	21
22	8.643	9.542	10.600	10.982	12.338	14.041	16.314	27.301	30.813	33.924	36.781	37.659	40.289	42.796	48.268	22
23	9.260	10.196	11.293	11.688	13.091	14.848	17.187	28.429	32.007	35.172	38.076	38.968	41.638	44.181	49.728	23
24	9.886	10.856	11.992	12.401	13.848	15.659	18.062	29.553	33.196	36.415	39.364	40.270	42.980	45.558	51.179	24
25	10.520	11.524	12.697	13.120	14.611	16.473	18.940	30.675	34.382	37.652	40.646	41.566	44.314	46.928	52.620	25
26	11.160	12.198	13.409	13.844	15.379	17.292	19.820	31.795	35.563	38.885	41.923	42.856	45.642	48.290	54.052	26
27	11.808	12.879	14.125	14.573	16.151	18.114	20.703	32.912	36.741	40.113	43.194	44.140	46.963	49.645	55.476	27
28	12.461	13.565	14.847	15.308	16.928	18.939	21.588	34.027	37.916	41.337	44.461	45.419	48.278	50.993	56.893	28
29	13.121	14.256	15.574	16.047	17.708	19.768	22.475	35.139	39.087	42.557	45.722	46.693	49.588	52.336	58.302	29
30	13.787	14.953	16.306	16.791	18.493	20.599	23.364	36.250	40.256	43.773	46.979	47.962	50.892	53.672	59.703	30
40	20.706	22.164	23.838	24.433	26.509	29.051	32.345	47.269	51.805	55.759	59.342	60.436	63.691	66.766	73.402	40
50	27.991	29.707	31.664	32.357	34.764	37.689	41.449	58.164	63.167	67.505	71.420	72.613	76.154	79.490	86.661	50
60	35.535	37.485	39.699	40.482	43.188	46.459	50.641	68.972	74.397	79.082	83.298	84.580	88.379	91.952	99.607	60
70	43.275	45.442	47.893	48.758	51.739	55.329	59.898	79.715	85.527	90.531	95.023	96.388	100.425	104.215	112.317	70
80	51.171	53.539	56.213	57.153	60.391	64.278	69.207	90.405	96.578	101.880	106.629	108.069	112.329	116.321	124.839	80
90	59.196	61.754	64.634	65.646	69.126	73.291	78.558	101.054	107.565	113.145	118.136	119.648	124.116	128.299	137.208	90
100	67.327	70.065	73.142	74.222	77.929	82.358	87.945	111.667	118.498	124.342	129.561	131.142	135.807	140.170	149.449	100

Figure 5: Chi square distribution table

Let χ^2_α be such that the area under the chi square distribution to its right is equal to α, see the tail in Figure 4. That is

$$P(\chi^2(\nu) \geq \chi^2_\alpha) = \alpha. \tag{24}$$

Example 8 *Let a random variable χ^2 have a chi square distribution with* 19 *degree of freedom, find $\chi^2_{0.05}$ and $\chi^2_{0.01}$.*

Solution. $\nu = 19$. By the chi square table (Figure 5),

$$\chi^2_{0.05} = 30.14, \quad \chi^2_{0.01} = 36.19.$$

Example 9 *Let $\alpha = 0.05$. Evaluate $P(\chi^2_{1-\alpha/2} < \chi^2(\nu) < \chi^2_{\alpha/2})$.*

Solution. Since

$$P(\chi^2(\nu) \geq \chi^2_{\alpha/2}) = \frac{\alpha}{2}, \quad P(\chi^2(\nu) \geq \chi^2_{1-\alpha/2}) = 1 - \frac{\alpha}{2},$$

$$P(\chi^2_{1-\alpha/2} < \chi^2(\nu) < \chi^2_{\alpha/2}) = 1 - \frac{\alpha}{2} - \frac{\alpha}{2} = 1 - 0.05 = 0.95.$$

(draw a plot for the example)

2.2 Representation of Chi Square Random Variable

Theorem 3 *Let $Z_1, Z_2, \ldots, Z_\nu$ be* independent *standard normal random variables, where ν is a positive integer. Then the random variable*

$$\chi^2(\nu) = \sum_{i=1}^{\nu} Z_i^2 \tag{25}$$

has a chi square distribution with ν degrees of freedom.

The sum of two independent chi square variables, $\chi^2(\nu_1) + \chi^2(\nu_2)$, has chi square distribution with degrees of freedom of $\nu_1 + \nu_2$.

Example 10 *If Z has a standard normal distribution, find the probability $P(Z^2 > 7.879)$.*

Solution. By Theorem 3, Z^2 has a chi square distribution with one degree of freedom. Therefore, by the chi square distribution table, we find $P(Z^2 > 7.879) = 0.005$.

Example 11 *Let $X_1, X_2 \ldots, X_n$ be independent normal random variables all having the same mean μ and variance σ^2. Define*

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{26}$$

and

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}.$$

Show that

(a) *Z has a standard normal distribution.*

(b) *Z^2 has a chi square distribution with one degree of freedom.*

(c) *Find $P(Z^2 > 5.024)$. Does this value depend on n, μ, or σ^2? Why?*

Solution.

(a). By Theorem 2, Z is a standard normal random variable.

(c). by Theorem 3, Z^2 has a chi square distribution with one degree of freedom.

(c). $\nu = 1$. By the chi square distribution table, $P(Z^2 > 5.024) = 0.025$.

Example 12 *Let $Z_1, Z_2, \ldots, Z_n$ be independent standard normal distribution. Show that*

(a)

$$\sum_{i=1}^{n} Z_i^2 = \sum_{i=1}^{n} (Z_i - \bar{Z})^2 + n\bar{Z}^2. \tag{27}$$

(b) *$\sqrt{n}\bar{Z}$ is a standard normal random variable;*

(c) *$n\bar{Z}^2$ has a chi square distribution with one degree of freedom;*

(d) *assume that the two terms on the right-hand side of (27) are independent, so* $\sum_{i=1}^n (Z_i - \bar{Z})^2$ *has a chi square distribution with* $n-1$ *degrees of freedom.*

Solution.

(a)

$$\sum_{i=1}^n Z_i^2 = \sum_{i=1}^n (Z_i - \bar{Z} + \bar{Z})^2 = \sum_{i=1}^n (Z_i - \bar{Z})^2 + 2\sum_{i=1}^n (Z_i - \bar{Z})\bar{Z} + n\bar{Z}^2 = \sum_{i=1}^n (Z_i - \bar{Z})^2 + n\bar{Z}^2.$$

(b). According to Theorem 2, $\bar{Z}$ is a normal random variable with mean 0 and variance $\frac{1}{n}$. Standardization gives

$$\sqrt{n}\bar{Z} = \frac{\bar{Z} - 0}{\sqrt{1/n}}.$$

Hence $\sqrt{n}\bar{Z}$ is a standard normal random variable.

(c). By Theorem 3, $n\bar{Z}^2$ has a chi square distribution with one degree of freedom.

(d). According to Theorem 3, $\sum_{i=1}^n Z_i^2$ has a chi square distribution with n degree of freedom. From part (c) above we have also known that $n\bar{Z}^2$ has a chi square distribution with one degree of freedom. Therefore, $\sum_{i=1}^n (Z_i - \bar{Z})^2$ has a chi square distribution with $n-1$ degrees of freedom.

Finally, if we define

$$S^2 = \frac{1}{n-1}\sum_{i=1}^n (Z_i - \bar{Z})^2, \tag{28}$$

we conclude from the previous example that $(n-1)S^2$ has a chi square distribution with $n-1$ degrees of freedom. That is, the scaling $\frac{1}{n-1}$ does not change the nature of the distribution, although it changes the specific form of the probability density function.

The above discussions can be generalized to the following result.

Theorem 4 *Let* $X_1, X_2, \ldots, X_n$ *be* independent *normal random variables all having the same mean* μ *and variance* σ^2*. Then*

$$\frac{(n-1)S^2}{\sigma^2} = \frac{1}{\sigma^2}\sum_{i=1}^n (X_i - \bar{X})^2 \tag{29}$$

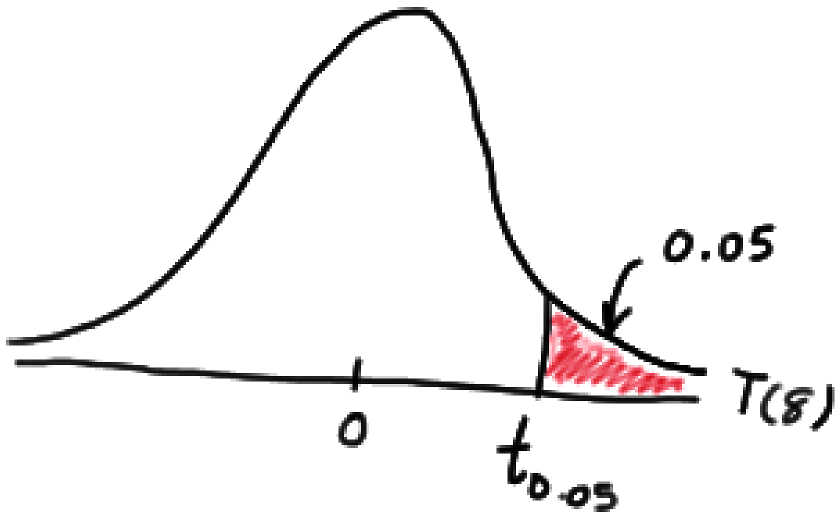


Figure 6: t-distribution with ν degrees of freedom, bell-shaped and symmetric

has a χ^2 distribution with $n-1$ degrees of freedom. Also, $\bar{X}$ and S^2 are independent random variables.

Interpretation of Theorem 4. Note that

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\bar{X})^2=\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}-\frac{\bar{X}-\mu}{\sigma}\right)^2.$$

Moreover,

$$\frac{X_i-\mu}{\sigma}\sim N(0,1),\quad \frac{\bar{X}-\mu}{\sigma}\sim N(0,\frac{1}{n}).$$

Consequently, $\frac{X_i-\mu}{\sigma}$ is like Z_i and $\frac{\bar{X}-\mu}{\sigma}$ is like $\bar{Z}$ in Example 12. So, by Eq. (28), $\frac{(n-1)S^2}{\sigma^2}$ has a χ^2 distribution with $n-1$ degrees of freedom.

2.3 Representation of t random variable

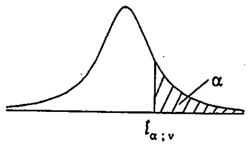
Definition 1 *Let the standard normal variable Z and chi square variable χ^2 defined in Eq. (25) be independent. Then*

$$t=\frac{\text{standard normal}}{\sqrt{\frac{\text{chi square}}{\text{degrees of freedom}}}}=\frac{Z}{\sqrt{\frac{\chi^2(\nu)}{\nu}}}=\frac{Z}{\sqrt{\frac{\sum_{i=1}^{\mu}Z_i^2}{\nu}}} \tag{30}$$

is said to have a t-distribution with v degrees of freedom.

Table of the Student's *t*-distribution

The table gives the values of $t_{\alpha;\nu}$ where
$\Pr(T_\nu > t_{\alpha;\nu}) = \alpha$, with ν degrees of freedom



ν \ α	0.1	0.05	0.025	0.01	0.005	0.001	0.0005
1	3.078	6.314	12.076	31.821	63.657	318.310	636.620
2	1.886	2.920	4.303	6.965	9.925	22.326	31.598
3	1.638	2.353	3.182	4.541	5.841	10.213	12.924
4	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	1.319	1.714	2.069	2.500	2.807	3.485	3.767
24	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	1.303	1.684	2.021	2.423	2.704	3.307	3.551
60	1.296	1.671	2.000	2.390	2.660	3.232	3.460
120	1.289	1.658	1.980	2.358	2.617	3.160	3.373
∞	1.282	1.645	1.960	2.326	2.576	3.090	3.291

Figure 7: t-distribution

We use t to denote a random variable having a t-distribution.

Let t_α be such that the area under the t-distribution curve to its right is equal to α, see the tail in Figure 6. That is

$$P(t \geq t_\alpha) = \alpha.$$

Example 13 *Let a random variable t have a t distribution with 19 degree of freedom, find $t_{0.05}$.*

Solution. $\nu = 19$. By the t-distribution table (Figure 7),

$$t_{0.05} = 1.729.$$

Example 14 *Let a random variable t have a t distribution with 19 degree of freedom, find $F(-1.729)$.*

Solution. Because t-distribution is **symmetric** (Figure 6),

$$F(-1.729) = P(t \leq -1.729) = P(t \geq 1.729).$$

By the previous example, for $\nu = 19$, $P(t \geq 1.729) = 0.05$. Therefore, $F(-1.729) = 0.05$.

Example 15 *Let $Z_1, \ldots, Z_6$ be independent each having a standard normal distribution. What is the distribution of the random variable*

$$\frac{Z_1 - Z_2}{\sqrt{\frac{Z_3^3 + Z_4^2 + Z_5^2 + Z_6^2}{2}}}?$$

Solution. Firstly, since Z_1 and Z_2 are independent standard normal random variables, by the property

$$Var(a_1 X_1 + a_2 X_2) = a_1^2 \sigma_{X_1}^2 + a_2^2 \sigma_{X_2}^2, \tag{31}$$

(see Eq. (11)) we find that the random variable

$$\frac{Z_1 - Z_2}{\sqrt{2}}$$

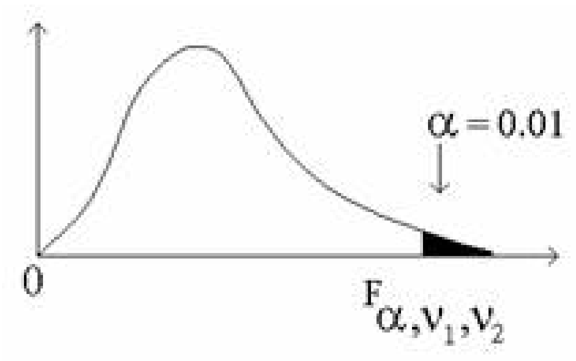


Figure 8: F distribution with ν_1 and ν_2 degrees of freedom

is a standard norm variable. Secondly, note that

$$\frac{Z_1 - Z_2}{\sqrt{\frac{Z_3^3+Z_4^2+Z_5^2+Z_6^2}{2}}} = \frac{\frac{Z_1-Z_2}{\sqrt{2}}}{\sqrt{\frac{Z_3^3+Z_4^2+Z_5^2+Z_6^2}{4}}}$$

Therefore by Eq. (30), $\frac{Z_1-Z_2}{\sqrt{\frac{Z_3^3+Z_4^2+Z_5^2+Z_6^2}{2}}}$ has t distribution with 4 degrees of freedom.

2.4 Representation of F random variables

Definition 2 *Let the chi square variables $\chi^2(\nu_1)$, with ν_1 degrees of freedom, and $\chi^2(\nu_2)$, with ν_2 degrees of freedom, be independent. Then*

$$F(\nu_1, \nu_2) = \frac{\frac{\text{chi square}}{\text{degrees of freedom}}}{\frac{\text{chi square}}{\text{degrees of freedom}}} = \frac{\frac{\chi^2(\nu_1)}{\nu_1}}{\frac{\chi^2(\nu_2)}{\nu_2}} = \frac{\frac{\sum_{i=1}^{\nu_1} Z_i^2}{\nu_1}}{\frac{\sum_{i=\nu_1+1}^{\nu_1+\nu_2} Z_i^2}{\nu_2}} \tag{32}$$

is said to have an F distribution with (ν_1, ν_2) degrees of freedom.

We use $F(\nu_1, \nu_2)$ to denote a random variable having an F-distribution with (ν_1, ν_2) degrees of freedom .

An important identity. Given $0 \leq \alpha \leq 1$, Let $F_\alpha(\nu_1, \nu_2)$ be such that the area under the F-distribution (with degrees of freedom (ν_1, ν_2)) curve to its right is equal to α, see the tail in Figure 8. Then

$$F_{1-\alpha}(\nu_1, \nu_2) = \frac{1}{F_\alpha(\nu_2, \nu_1)}. \tag{33}$$

Example 16 *Find the value of $F_{0.95}(10, 20)$.*

Table 6(a) Values of $F_{0.05}$

ν_2 = Degrees of Freedom for Denominator	ν_1 = Degrees of Freedom for Numerator																		
	1	2	3	4	5	6	7	8	9	10	12	15	20	25	30	40	60	120	∞
1	161	200	216	225	230	234	237	239	241	242	244	246	248	249	250	251	252	253	254
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.46	19.46	19.47	19.48	19.49	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.63	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.52	4.50	4.46	4.43	4.40	4.37
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.83	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.40	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.11	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.89	2.86	2.83	2.79	2.75	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.73	2.70	2.66	2.62	2.58	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.60	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.50	2.47	2.38	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.41	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.34	2.31	2.27	2.22	2.18	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.28	2.25	2.20	2.16	2.11	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.23	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.18	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.14	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.07	2.04	1.99	1.95	1.90	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.02	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.00	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.97	1.94	1.89	1.84	1.79	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.88	1.84	1.79	1.74	1.68	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.78	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.69	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.60	1.55	1.50	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.51	1.46	1.39	1.32	1.22	1.00

Table 6(b) Values of $F_{0.01}$

ν_2 = Degrees of Freedom for Denominator	ν_1 = Degrees of Freedom for Numerator																		
	1	2	3	4	5	6	7	8	9	10	12	15	20	25	30	40	60	120	∞
1	4,052	5,000	5,403	5,625	5,764	5,859	5,928	5,982	6,023	6,056	6,106	6,157	6,209	6,240	6,261	6,287	6,313	6,339	6,366
2	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39	99.40	99.42	99.43	99.45	99.46	99.57	99.47	99.48	99.49	99.50
3	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35	27.23	27.05	26.87	26.69	26.58	26.50	26.41	26.32	26.22	26.13
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55	14.37	14.20	14.02	13.91	13.84	13.75	13.65	13.56	13.46
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05	9.89	9.72	9.55	9.45	9.38	9.29	9.20	9.11	9.02
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.56	7.40	7.30	7.23	7.14	7.06	6.97	6.88
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.31	6.16	6.06	5.99	5.91	5.82	5.74	5.65
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.52	5.36	5.26	5.20	5.12	5.03	4.95	4.86
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11	4.96	4.81	4.71	4.65	4.57	4.48	4.40	4.31
10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.56	4.41	4.31	4.25	4.17	4.08	4.00	3.91
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.40	4.25	4.10	4.01	3.94	3.86	3.78	3.69	3.60
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.01	3.86	3.76	3.70	3.62	3.54	3.45	3.36
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	3.96	3.82	3.66	3.57	3.51	3.43	3.34	3.25	3.17
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03	3.94	3.80	3.66	3.51	3.41	3.35	3.27	3.18	3.09	3.00
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67	3.52	3.37	3.28	3.21	3.13	3.05	2.96	2.87
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.55	3.41	3.26	3.16	3.10	3.02	2.93	2.84	2.75
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.46	3.31	3.16	3.07	3.00	2.92	2.83	2.75	2.65
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.37	3.23	3.08	2.98	2.92	2.84	2.75	2.66	2.57
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.30	3.15	3.00	2.91	2.84	2.76	2.67	2.58	2.49
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.23	3.09	2.94	2.84	2.78	2.69	2.61	2.52	2.42
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	3.17	3.03	2.88	2.79	2.72	2.64	2.55	2.46	2.36
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.12	2.98	2.83	2.73	2.67	2.58	2.50	2.40	2.31
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	3.07	2.93	2.78	2.69	2.62	2.54	2.45	2.35	2.26
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.03	2.89	2.74	2.64	2.58	2.49	2.40	2.31	2.21
25	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22	3.13	2.99	2.85	2.70	2.60	2.54	2.45	2.36	2.27	2.17
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.84	2.70	2.55	2.45	2.39	2.30	2.21	2.11	2.01
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.66	2.52	2.37	2.27	2.20	2.11	2.02	1.92	1.80
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.50	2.35	2.20	2.10	2.03	1.94	1.84	1.73	1.60
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.34	2.19	2.03	1.93	1.86	1.76	1.66	1.53	1.38
∞	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.18	2.04	1.88	1.77	1.70	1.59	1.47	1.32	1.00

Figure 9: F distribution

Solution. By Eq. (33)

$$F_{0.95}(10, 20) = \frac{1}{F_{0.05}(20, 10)}.$$

Using the F table (Figure 9) to get

$$\frac{1}{F_{0.05}(20, 10)} = \frac{1}{2.77} = 0.3610.$$

Therefore, $F_{0.95}(10, 20) = 0.3610$.

Example 17 *Let t be distributed as a t-distribution with ν degrees of freedom.*

(a) *Use the representation of t to show that t^2 has an F distribution with $(1, \nu)$ degrees of freedom.*

(b) *Use part (a) to show that $t^2_{\alpha/2} = F_\alpha(1, \nu)$.*

Solution.

(a). By Eq. (30), assume t is in the form of

$$t = \frac{Z}{\sqrt{\frac{\chi^2(\nu)}{\nu}}},$$

where $Z \sim N(0, 1)$ and $\chi^2(\nu)$ has a chi square distribution of ν degrees of freedom. So Z^2 has a chi square distribution of one degree of freedom. Consequently by Definition 2, t^2 has an F distribution with $(1, \nu)$ degrees of freedom.

(b). Since

$$P(t > t_{\alpha/2}) = \frac{\alpha}{2},$$

we have (note that t-distribution is symmetric)

$$P(t^2 \leq t^2_{\alpha/2}) = 1 - \alpha.$$

On the other hand, by part (a), t^2 has an F distribution with $(1, \nu)$ degrees of freedom. Therefore

$$P(t^2 \leq F_\alpha(1, \nu)) = 1 - P(t^2 \geq F_\alpha(1, \nu)) = 1 - \alpha.$$

Therefore, $t^2_{\alpha/2} = F_\alpha(1, \nu)$.