MATHEMATICS 3103 (Functional Analysis)
YEAR 2012–2013, TERM 2

PROBLEM SET #7

This problem set is due at the *beginning* of class on Thursday 21 March. Only Problem 1 will be formally assessed; but I think you will find the other problems (especially #3 and #5) useful in clarifying the meaning and applications of the uniform boundedness and closed graph theorems.

Topics: The Baire category theorem and its applications. The uniform boundedness theorem, the open mapping theorem, the closed graph theorem.

Readings:

- Handout #7: The Baire category theorem and its consequences.

1. **A positive application of the uniform boundedness theorem.** Fix $p \in [1, \infty]$, and define q as usual by $1/p + 1/q = 1$. Now let $(a_k)_{k=1}^{\infty}$ be a sequence of real numbers with the property that $\sum\limits_{k=1}^{\infty} a_k x_k = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} a_k x_k$ exists (and is finite) for all $x \in \ell^p$. Prove that $a \in \ell^q$ (so that the sum is in fact absolutely convergent). [*Hint:* Consider the linear functionals $\varphi_n(x) = \sum\limits_{k=1}^{n} a_k x_k$ on ℓ^p and apply the uniform boundedness theorem. In the case $p = \infty$ you can even restrict to $x \in c_0$.]

2. **A negative application of the uniform boundedness theorem.** It follows from our Hilbert-space theory combined with Theorem 5.15 that the Fourier series of any function $f \in \mathcal{C}[-\pi, \pi]$ (or even $f \in L^2[-\pi, \pi]$) is convergent in L^2 norm to f. But must it converge *pointwise*? The answer is no: in 1876 du Bois-Reymond constructed an example of a continuous function whose Fourier series is divergent at 0. Using the uniform boundedness theorem you can give an easy (albeit nonconstructive) proof of the existence of such a function.

   Recall that the Fourier series for a function f on $[-\pi, \pi]$ is $\sum\limits_{n=-\infty}^{\infty} a_n e^{inx}$, where the Fourier coefficients are given by

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} f(x)\, dx \qquad \text{for } n \in \mathbb{Z}\,.$$

(I am using here the complex form of the Fourier series, which in my opinion is more convenient. But if you prefer to use instead the real form involving sines and cosines, that is fine too.) So the Nth partial sum of the Fourier series is, by definition,

$$(S_N f)(x) \;=\; \sum_{n=-N}^{N} a_n \, e^{inx}$$

where the $\{a_n\}$ are as above.

(a) Prove that

$$(S_N f)(x) \;=\; \frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(x-x') \, f(x') \, dx'$$

where

$$K_N(t) \;=\; \frac{\sin(N+\frac{1}{2})t}{\sin \frac{1}{2}t} \;.$$

[In doing this calculation, be careful not to confuse the point x where $S_N f$ is being evaluated with the integration variable arising in the definition of a_n: call the latter x'.]

(b) Define a linear functional φ_N on $\mathcal{C}[-\pi,\pi]$ by $\varphi_N(f) = (S_N f)(0)$. Prove that φ_N is a bounded linear functional, of norm

$$\|\varphi_N\| \;=\; \frac{1}{2\pi} \int_{-\pi}^{\pi} |K_N(t)| \, dt \;.$$

[*Hint:* The proof that $\|\varphi_N\|$ is bounded *above* by this quantity is quite easy. To show that $\|\varphi_N\|$ is bounded *below* by this quantity, consider a continuous f that approximates $\operatorname{sgn} K_N(t)$. You can do this either "by your bare hands", or by using Urysohn's lemma with the sets $A = \{t\colon\, K_N(t) \ge \epsilon\}$ and $B = \{t\colon\, K_N(t) \le -\epsilon\}$.]

(c) Show that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |K_N(t)| \, dt \;\ge\; \frac{4}{\pi^2} \sum_{j=0}^{2N} \frac{1}{j+1} \;\ge\; \frac{4}{\pi^2} \log(2N+2) \;.$$

(d) Invoke the uniform boundedness theorem to conclude that the family (φ_N) of linear functionals cannot be pointwise bounded on $\mathcal{C}[-\pi,\pi]$ — or in other words, that there exists $f \in \mathcal{C}[-\pi,\pi]$ such that the sequence $((S_N f)(0))$ is unbounded.

3. **Strong, weak and weak-* boundedness.** As you know, a subset A of a normed linear space X is called **bounded** (or **norm-bounded** or **strongly bounded**) if $\sup\limits_{x \in A} \|x\| < \infty$. Let us now say that a subset $A \subseteq X$ is **weakly bounded** if $\sup\limits_{x \in A} |\ell(x)| < \infty$ for all $\ell \in X^*$. Finally, let us say that a subset $B \subseteq X^*$ is **weak-* bounded** if $\sup\limits_{\ell \in B} |\ell(x)| < \infty$ for all $x \in X$.

(a) Prove that a set A in a normed linear space X is bounded if and only if it is weakly bounded.

(b) Prove that if X is complete, then a set B in X^* is bounded if and only if it is weak-* bounded.

(c) Give an example of an incomplete normed linear space X and a subset $B \subseteq X^*$ that is weak-* bounded but not bounded.

[*Hint:* For (a), use the uniform boundedness theorem together with the natural embedding of X into X^{**}. For (b), use the uniform boundedness theorem. For (c), use an example showing that the conclusion of the uniform boundedness theorem can fail if X is incomplete.]

4. **An extension of the uniform boundedness theorem.** Let X and Y be normed linear spaces, and let $\mathcal{F} \subseteq \mathcal{B}(X, Y)$ be a family of bounded linear maps from X to Y. We say that the family $\mathcal{F}$ is

   - **bounded** (or **uniformly bounded**) if $\sup\limits_{T \in \mathcal{F}} \|T\|_{X \to Y} < \infty$;
   - **pointwise bounded** if $\sup\limits_{T \in \mathcal{F}} \|Tx\|_Y < \infty$ for all $x \in X$;
   - **weakly pointwise bounded** if $\sup\limits_{T \in \mathcal{F}} |\ell(Tx)| < \infty$ for all $x \in X$ and $\ell \in Y^*$.

   The uniform boundedness theorem, as proven in class, states that if X is a Banach space, then the family $\mathcal{F}$ is bounded if and only if it is pointwise bounded. I would like you to prove now that these conditions are also equivalent to weak pointwise boundedness. [*Hint:* For each $x \in X$, consider the family $\{\widehat{Tx}\colon T \in \mathcal{F}\}$ in Y^{**}, where $y \mapsto \widehat{y}$ denotes the natural embedding of Y into Y^{**}, and apply the uniform boundedness theorem. Alternatively, apply part (a) of the preceding problem.]

5. Let X and Y be Banach spaces, and let $T\colon X \to Y$ be an *injective* bounded linear map. Prove that the inverse map $T^{-1}\colon T[X] \to X$ is bounded if and only if $T[X]$ is closed in Y.