LMS-Bath Symposium
Advances in Spectral Theory

List of abstracts and titles

22–26 July 2024

Rafael Benguria *(Pontificia Universidad Catolica de Chile)*
Title: Estimates on the L^2 norm of the positive solutions of a two parameter family of nonlinear PDE´s of the TFW type.
Abstract: In this talk I consider the two parameter family of PDE´s (generalized TFW equation):

$$-\Delta u + (\gamma\, u^{2p-2} - \phi)u = 0$$

on $\mathbb{R}^3$, where,

$$\phi(x) = \frac{Z}{|x|} - \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|}\, dy.$$

Here, Z is fixed and $\gamma \geq 0$, and $p > 1$. The case $p = 5/3$, for any $\gamma > 0$ corresponds to the Thomas–Fermi–von Weizsäcker equation in the atomic case, which was first studied by R. Benguria, H. Brezis, and E.H. Lieb in 1981. The case $\gamma = 0$ corresponds to the Hartree equation. Here we are interested in estimating the so called "excess charge", given by $Q = ||u||_2^2 - Z$, for different values of p and γ , where u is the positive solution of the generalized TFW equation. This is joint work with Heinz Siedentop (Ludwig Maximilians University).

Michiel van den Berg *(University of Bristol)*
Title: On the torsion function for simply connected, open sets in $\mathbb{R}^2$.
Abstract: For an open set $\Omega \subset \mathbb{R}^2$ let $\lambda(\Omega)$ denote the bottom of the spectrum of the Dirichlet Laplacian acting in $L^2(\Omega)$. Let w_Ω be the torsion function for Ω, and let $\|.\|_p$ denote the L^p norm. It is shown there exists $\eta > 0$ such that $\|w_\Omega\|_\infty \lambda(\Omega) \geq 1 + \eta$ for any non-empty, open, simply connected set $\Omega \subset \mathbb{R}^2$ with $\lambda(\Omega) > 0$. Joint work with Dorin Bucur.

Sabine Bögli *(Durham University)*
Title: On the discrete eigenvalues of Schrödinger operators with complex potentials
Abstract: In this talk I shall present constructions of Schrödinger operators with complex-valued potentials whose spectra exhibit interesting properties. One example shows that for sufficiently large p, the discrete eigenvalues need not be bounded in modulus by the L^p norm of the potential. This is a counterexample

to the Laptev-Safronov conjecture (Comm. Math. Phys. 2009). Another construction proves optimality (in some sense) of generalisations of Lieb-Thirring inequalities to the non-selfadjoint case - thus giving us information about the accumulation rate of the discrete eigenvalues to the essential spectrum.

This talk is based on joint works with Jean-Claude Cuenin (Loughborough) and Frantisek Stampach (Prague).

Jade Brisson *(Université de Neuchâtel)*

Title: Upper bounds, spectral ratios and spectral gaps for Steklov eigenvalues of warped products

Abstract: In the first part of the talk, we investigate the Steklov spectrum of the warped product $[0, L] \times_h \Sigma$ equipped with the metric $dt^2 + h(t)^2 g_\Sigma$, where Σ is a compact surface. We find sharp upper bounds for the Steklov eigenvalues in terms of the eigenvalues of the Laplacian on Σ. In particular, we apply our method to the case of metric of revolution on the 3-dimensional ball and we obtain a sharp estimate on the spectral gap between two consecutive Steklov eigenvalues.

In the second part, we investigate the spectral ratios as well as spectral gaps for higher order Steklov eigenvalues of Riemannian manifolds with revolution-type metrics. This is based on joint works with Bruno Colbois and Katie Gittins.

Matteo Capoferri *(Heriot-Watt University)*

Title: The operator curl: a case study in the spectral theory of systems. I

Abstract: In my talk I will present a new approach to the spectral theory of systems of PDEs on closed manifolds, developed in a series of recent papers by Dmitri Vassiliev (UCL) and myself, based on the use of pseudodifferential projections. After discussing the general theory, I will turn to the (non-elliptic) operator curl, and explain how our techniques offer a new pathway to the study of spectral asymmetry.

[1] Invariant subspaces of elliptic systems I: pseudodifferential projections. Journal of Functional Analysis 282 (2022) 109402. [2] Invariant subspaces of elliptic systems II: spectral theory. Journal of Spectral Theory 12 (2022) 301-338. [3] Beyond the Hodge Theorem: curl and asymmetric pseudodifferential projections. Preprint arXiv:2309.02015.

Alex Cohen *(Massachusetts Institute of Technology)*

Title: Fractal uncertainty in higher dimensions

Abstract: A fractal uncertainty principle (FUP) roughly says that a function and its Fourier transform cannot both be concentrated on a fractal set. These were introduced to harmonic analysis in order to prove new results in quantum chaos: if eigenfunctions on hyperbolic manifolds concentrated in unexpected ways, that would contradict the FUP. Bourgain and Dyatlov proved FUP over the real numbers, and in this talk I will discuss an extension to higher dimensions. The bulk of the work is constructing certain plurisubharmonic functions on $\mathbb{C}^n$.

Jean-Claude Cuenin *(Loughborough University)*

Title: Spectral cluster bounds for orthonormal functions on compact manifolds with boundary

Abstract: I will talk about L^q bounds for spectral clusters (linear combinations of eigenfunctions) of the Laplace-Beltrami operator on a 2-dimensional compact manifold with boundary. Smith and Sogge (2007) proved optimal bounds for the case of a single cluster. In this talk, we will consider N orthonormal clusters and seek bounds on the L^q norm of their square function (or density). This program was initiated by Frank and Sabin (2017), who extended the classical L^q bounds of Sogge (1988) in the boundaryless case. The main challenge is to obtain an optimal dependence on N (the number of functions involved). The talk is based on joint work with Xiaoyan Su.

David Damanik *(Rice University)*

Title: Existence of Nonuniform Cocycles

Abstract: Peter Walters asked in 1986 whether any uniquely ergodic homeomorphism with a non-atomic invariant probability measure admits a non-uniform GL(2,R) cocycle. We discuss the history and context of this question and then present joint work with Artur Avila that gives an affirmative answer to it.

Alix Deleporte *(Université Paris-Saclay)*

Title: Semiclassical analysis of free fermions

Abstract: To each orthogonal projector of finite rank N on $L^2(R^d)$ is associated a point process on R^d with N points, which gives the joint probability density of fermions that fill the image of the projector.

The study of the statistical properties of these fermions, in the large N limit, is linked to semiclassical spectral theory problems, some of them well studied (the Weyl law gives a law of large numbers), some of them new. In particular, the behaviour of the variance is linked with the properties of commutators involving spectral projectors, which are not so well understood.

In this talk, I will present my work in collaboration with Gaultier Lambert (KTH) on this topic.

Véronique Fischer *(University of Bath)*

Title: Towards quantum limits for subelliptic operators

Abstract: The aim of the talk is to present recent developments of high frequency analysis for sub-elliptic operators and in sub-Riemannian geometry. I will start with discussing why these questions are closely related to many aspects of harmonic analysis.

Rupert Frank *(Ludwig-Maximilians-Universität München)*

Title: Riesz means asymptotics for Laplacians on Lipschitz domains

Abstract: We consider the eigenvalues of the Dirichlet and Neumann Laplacians on a bounded domain with Lipschitz boundary and prove two-term asymptotics for their Riesz means of arbitrary positive order. Moreover, when the underlying domain is convex, we obtain a universal, non-asymptotic bound that correctly

reproduces the two leading terms in the asymptotics and depends on the domain only through simple geometric characteristics.

Asma Hassannezhad *(University of Bristol)*

Title: Steklov eigenvalues of hyperbolic manifolds with totally geodesic boundary

Abstract: The geometry and topology of closed negatively curved manifolds are subtly reflected in a geometric bound for the Laplace eigenvalues. In 1980, Schoen, Wolpert, and Yau showed that the small Laplace eigenvalues can be bounded from below and above by the length of a shortest multi-geodesics dividing the surface into disjoint connected components. Schoen later obtained a spectral gap on negatively curved manifolds in higher dimensions which is in contrast with the result for hyperbolic surfaces. In this talk, we discuss how these results can be extended to the setting of the Steklov eigenvalue problem.

Mikhail Karpukhin *(University College London)*

Title: New embedded minimal surfaces in 3-sphere and 3-ball via eigenvalue optimisation

Abstract: The study of optimal upper bounds for Laplace eigenvalues on closed surfaces under area constraint is a classical problem of spectral geometry. It is particularly interesting due to the fact that optimal metrics (if exist) correspond to branched minimal surfaces in n-dimensional sphere. In general, determining whether such metrics exist, whether the corresponding maps are embeddings, and determining the dimension of the sphere are challenging problems, where very few results are known. In the present talk we will discuss how one can use group action to resolve these issues and, as a result, construct many new examples of embedded minimal surfaces in the 3-sphere. The same considerations can be applied to the Steklov eigenvalue problem. As a consequence, we completely resolve the realisation problem for free boundary minimal surfaces in the unit 3-ball: we show that any compact orientable surface with boundary can be embedded in the 3-ball as a free boundary minimal surface. Based on a joint work with R. Kusner, P. McGrath and D. Stern.

Gerasim Kokarev *(University of Leeds)*

Title: Laplacian spectra of minimal submanifolds in the hyperbolic space

Abstract: I will describe an extremal problem for the fundamental tone of submanifolds in the hyperbolic space, and will show that singular minimal submanifolds occur as natural maximisers for it. I will also discuss closely related computations of the Laplacian spectra of minimal submanifolds.

Igor Krasovsky *(Imperial College London)*

Title: Probability of two large gaps in the bulk of the spectrum of random matrices

Abstract: For many ensembles of random matrices, the eigenvalues in the bulk of the spectrum follow the sine-kernel point process. We are interested in the probability of several intervals without eigenvalues (gaps) when the intervals are

large. Following the work of Dyson, Mehta, Widom, the asymptotic formula for this probability was derived in a classical work of Deift, Its, and Zhou in 1997. However, the constant term in the asymptotics remained undetermined. In the case of two or more gaps, oscillations described by the theta-functions appear in the asymptotics complicating the analysis. Several years ago in a joint work with Benjamin Fahs, we managed to find the constant term for two gaps. In the talk, this result and some history will be reviewed.

Fabricio Macià *(Universidad Politécnica de Madrid)*
Title: Born approximation and Dirichlet-to-Neumann maps
Abstract: We will focus on the inverse problem of recovering a potential of a Schrödinger operator from the knowledge of its Dirichlet-to-Neumann map. This is known as the Calderón problem, or Electric Impedance Tomography, and has been intensively studied in the past forty years. This inverse problem is severely ill-posed, which makes the task of designing efficient algorithms to solve it particularly difficult. We will rigorously prove, in the simplified setting of radial potentials in the euclidean unit ball, the existence of the Born approximation, a function that encodes the whole DtN map and enjoys several interesting qualitative and quantitative approximation properties. We use this function to factorize the inverse problem into a linear (ill-posed but explicit) and a nonlinear (well-posed, Hölder continuous) part. This factorization gives a (partial) characterization of the set of DtN maps and we will show how this can be used to ultimately design efficient algorithms to solve the inverse problem. Our analysis is based on results on inverse spectral theory for Schrödinger operators on the half-line, in particular on the concept of A-amplitude introduced by Barry Simon in 1999

Marco Marletta *(Cardiff University)*
Title: Computing spectra of perturbed periodic operators: how hard can it be?
Abstract: We consider Schr¨odinger operators in $\mathbb{R}^d$ or on waveguides, with complex-valued potentials that are either periodic or peturbations of periodic. The periodic part of the potential is allowed to be non-real. We try to answer the question of the title from two standpoints:

1. the Hansen SCI (solvability complexity index);

2. actually doing numerics, seeing how long they take and how good or bad the results may be.

From the standpoint of SCI we show that, for pure periodic problems, there is no difference between the selfadjoint and non-selfadjoint cases, even though in the latter case it is very easy to construct examples whose spectrum contains non-empty open sets in $\mathbb{C}$.

This talk is based on published joint works with Aljawi [IEOT 2023] and with Ben-Artzi and R¨osler [Numerische Math 2022]. I shall also show some very recent numerics by Lucie Levadoux.

Ngoc Nhi Nguyen *(University of Milan)*
Title: Spectral cluster bounds for orthonormal functions on compact manifolds with non-smooth metrics
Abstract: There has been substantial recent interest in functional inequalities for systems of orthonormal functions. The game is to prove an optimal dependence on the number of functions involved. In this talk, we focus on a family of inequalities called "spectral cluster bounds", which concern L^p norms of (linear combinations of) eigenfunctions of the Laplace-Beltrami operator on a compact closed manifold. Since the seminal work of Sogge in the 1980's, these bounds have been generalized in various directions. Frank and Sabin recently established a version of Sogge's bounds for systems of orthonormal functions. The result is valid for smooth metrics. We will show that the same result holds for C^s metrics. The analogue in the one-function case was proved by Smith and Koch, Smith and Tataru. The talk is based on joint ongoing work with Jean-Claude Cuenin.

Stéphane Nonnenmacher *(Université Paris-Saclay)*
Title: Random eigenstates of the Quantum Cat Map
(Joint with Nir Schwartz)
Abstract: Long standing conjectures in Quantum Chaos concerns the equidistribution and statistical properties of eigenstates of quantized chaotic systems, in the semiclassical regime: one expects Quantum Unique Ergodicity (all eigenmodes should asymptotically equidistribute), and at the microscopic scale, the eigenmodes are expected to satisfy the same statistical properties as random monochromatic waves (Berry's random wave model).

So far, results on the latter conjecture have been obtained for random quasimodes of chaotic systems (e.g. the Laplacian on a closed manifold of negative sectional curvature), in certain spectral windows. One difficulty comes from the fact that for these systems, eigenvalues are expected to be nondegenerate, or weakly degenerate.

The quantized hyperbolic automorphisms of the 2-torus, also known as "Quantum Cat Maps", form a toy model of quantized chaotic systems enjoying specific properties, wich allows to investigate more thoroughly their spectra. A peculiarity of this model is the presence, in the semiclassical limit, of "maximally large" spectral multiplicities (in the context of hyperbolic surfaces, such multiplicities would saturate Bérard's bound).

These large multiplicities allow us to consider random eigenbases of the Quantum Cat Map. We prove that, with high probability, those random eigenbases satisfy Quantum Unique Ergodicity, including at (algebraic) small scales. We also show that the local statistics of these random eigenstates converges to those of standard Gaussian random states, the analogue of Berry's random wave model.

Leonid Pastur *(King's College London and Institute for Low Temperature Physics and Engineering, Kharkiv, Ukraine)*
Title: Quantum Entanglement and Spectral Theory
Abstract: We give an overview of recent results and problems associated with

asymptotic trace formulas that arise when studying the large-size behaviour of the entanglement entropy of extended quantum systems, mainly the free-fermions models, both translation invariant and disordered.

Julie Rowlett *(Chalmers University of Technology)*
Title: The L^p Laplace spectrum on conformally compact manifolds
Abstract: I will report on joint work with Nelia Charalambous. Conformally compact manifolds are a class of non-compact manifolds with variable curvature that were introduced by Fefferman and Graham to study conformal invariants. They are a broad class and include many examples, including manifolds with Poincaré-Einstein metrics. Motivation to study the Laplace operator acting on Lp for p not equal to 2 comes from physics. For example, the most natural space on which to study heat diffusion is L1. Here, we show that for general values of p, the Lp Laplace spectrum contains a certain parabolic region and is contained in a certain parabolic region. These regions are determined by the geometry of the conformally compact manifold and the value of p.

Grigori Rozenblum *(Chalmers University of Technology)*
Title: Eigenvalue asymptotics of singular measures and Connes' noncommutative integration
Abstract: In a domain $\Omega \subset \mathbb{R}^{\mathbf{N}}$ we consider compact, Birman-Schwinger type, operators of the form $\mathbf{T}_{P,\mathfrak{A}} = \mathfrak{A}^* P \mathfrak{A}$, where P is a Borel measure in Ω, singular with respect to the Lebesgue measure and $\mathfrak{A}$ is an order $-\mathbf{N}/2$ pseudodifferential operator. For a class of such operators, we obtain order sharp eigenvalue estimates and a proper version of H.Weyl's asymptotic law for eigenvalues, with order not depending on dimensional characteristics of the measure. These results lead to establishing measurability, in the sense of Dixmier - Connes, of such operators and the noncommutative version of integration over Lipschitz surfaces and some more general sets.

Reference: Eigenvalues of singular measures and Connes' noncommutative integration. J. Spectr. Theory ***12*** (2022), no. 1, 259–300.

Jeffrey Schenker *(Michigan State University)*
Title: Disordered Quantum Trajectories under Random Generalized Measurements
Abstract: A general framework to study the behavior of quantum trajectories obtained by repeated random measurements subject to not-necessarily-independent but stationary and ergodic noise will be discussed. Two results will be highlighted: 1) a large deviation principle for the results of random measurements and 2) a generalization to the disordered context of results of Kümmerer and Maassen on asymptotic purification to the ergodic, disordered setting

Iveta Semorádová *(Cardiff University)*
Title: $\mathcal{PT}$-symmetric oscillators with one-center point interactions
Abstract: It is well established that the Schrödinger operators with $\mathcal{PT}$-symmetric

polynomial potentials,

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - (ix)^n \tag{1}$$

defined on $L^2(R)$, possess real spectrum for $n \geq 2$, $n \in N$. Moreover, in the case $n = 1$, the spectrum of the imaginary Airy operator is empty.

For $n = 2k + 1$, $k \geq 0$, the spectra of such operators defined on $L^2(R_+)$ and $L^2(R_-)$, with Dirichlet or Neumann boundary conditions in the origin, consists only of non-real eigenvalues.

Perturbing the problem on $L^2(R)$ with delta interaction centered at the origin

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - (ix)^{2k+1} + \alpha\delta, \tag{2}$$

we observe $\mathcal{PT}$-symmetric phase transition, where the real eigenvalues transition into complex conjugate pairs, asymptotically converging to the eigenvalues of the problems on $L^2(R_+)$ and $L^2(R_-)$ with Dirichlet boundary conditions, as $|\alpha| \to \infty$. Moreover, for $\alpha < 0$, there is a real eigenvalue in the spectrum, which tends to $-\infty$ as $\alpha \to -\infty$. The first result follows also for the perturbation with delta prime interaction, where the eigenvalues asymptotically tend to those of the problems on $L^2(R_+)$ and $L^2(R_-)$ with Neumann boundary conditions.

Based on:
[1] Behrndt, Semorádová, Siegl. The imaginary Airy operator with one-center δ interaction. To appear in Pure and Applied Functional Analysis
[2] Marletta, Semorádová. PT-symmetric oscillators with one-center point interactions. Manuscript in preparation.

Mira Shamis *(Queen Mary, University of London)*

Title: On the abominable properties of the Almost Mathieu operator with Liouville frequencies

Abstract: This talk is devoted to the study of some spectral properties of the Almost Mathieu Operator: that is one-dimensional discrete Schroedinger operator that acts on the space of square-summable sequences as a sum of the free discrete Laplacian and multiplication by a potential of the form $\lambda * \cos(2\pi\alpha * n + \theta)$. The parameter α, called the frequency, is some number between zero and one. It is well-known that the spectral properties of the Almost Mathieu operator depend sensitively on the arithmetic properties of the frequency. The case of poorly approximated frequencies that satisfy a certain Diophantine condition, is relatively well understood. In that case the spectral properties are as nice as one would expect. There is a completely different picture in the case of well approximated frequencies (Liouville numbers), in which case we show that several spectral characteristics of the Almost Mathieu operator can be as poor as at all possible in the class of all discrete Schroedinger operators. For example, the modulus of continuity of the integrated density of states (that is, of the averaged spectral measure) may be no better than logarithmic. The logarithmic modulus of continuity of the integrated density of states is known to be the optimal modulus of continuity in the class of all discrete Shroedinger operators.

Other characteristics to be discussed are the Hausdorff measure of the spectrum for the so-called critical case when $\lambda = 1$, and non-homogeneity of the spectrum (as a set) for a range of λ-s. Based on joint work with A. Avila, Y. Last, and Q. Zhou

David Sher *(DePaul University)*

Title: Variations on Polya's conjecture for the disk

Abstract: I will give an overview of several recent results concerning the solution to Polya's conjecture for the disk. These include variations on the conjecture itself and also consequences for numerical approximation of Bessel functions. This is joint work with N. Filonov (St. Petersburg State), M. Levitin (Reading), and I. Polterovich (Montréal).

Nadia Sidorova *(University College London)*

Title: Localisation and delocalisation in the parabolic Anderson model

Abstract: The parabolic Anderson problem is the Cauchy problem for the heat equation on the integer lattice with random potential. It describes the mean-field behaviour of a continuous-time branching random walk. It is well-known that, unlike the standard heat equation, the solution of the parabolic Anderson model exhibits strong localisation. In particular, for a wide class of iid potentials it is localised at just one point. However, in a partially symmetric parabolic Anderson model, the one-point localisation breaks down for heavy-tailed potentials and remains unchanged for light-tailed potentials, exhibiting a range of phase transitions.

Mikhail Sodin *(Tel Aviv University)*

Title: Fourier Uniqueness and Nonuniqueness Pairs

Abstract: My talk is based on a joint work with A. Kulikov and F. Nazarov (arxiv.org/abs/2306.14013) and will concern the spectral theory of functions. Motivated by a discovery by Radchenko and Viazovska, and by work by Ramos and Sousa, we find conditions sufficient for a pair of discrete subsets of the real axis to be a uniqueness or a non-uniqueness pair for the Fourier transform. These conditions are not too far from each other. The uniqueness theorem can be upgraded to a quantitative one. If time permits, we will also discuss discrete uniqueness pairs in $\mathbb{R}^d$ with $d > 1$.

Dmitri Vassiliev *(University College London)*

Title: The operator curl: a case study in the spectral theory of systems. II

Abstract: This talk is a continuation of Matteo Capoferri's, promoting a new approach to the study of spectral asymmetry. Working with the operator curl $:= *\mathrm{d}$ on a connected oriented closed Riemannian 3-manifold, we construct, by means of microlocal analysis, the asymmetry operator — a scalar pseudodifferential operator of order -3. The latter is completely determined by the Riemannian manifold and its orientation, and encodes information about spectral asymmetry. The asymmetry operator generalises and contains the classical eta invariant traditionally associated with the asymmetry of the spectrum, which

can be recovered by computing its regularised operator trace. Remarkably, the whole construction is direct and explicit.

[1] Beyond the Hodge Theorem: curl and asymmetric pseudodifferential projections. Preprint arXiv:2309.02015."

Ian Wood *(University of Kent)*

Title: Spectrum of the Maxwell Equations for a Flat Interface between Homogeneous Dispersive Media

Abstract: In this talk, we consider the spectrum of a non-selfadjoint operator pencil generated by the time-harmonic Maxwell problem with a nonlinear dependence on the frequency. We look in detail at the case of a one-dimensional reduction for the situation of two homogeneous materials joined at a planar interface. The dependence on the spectral parameter, i.e. the frequency, is in the dielectric function and we make no assumptions on its form. In order to allow also for non-conservative media, the dielectric function is allowed to be complex, yielding a non-selfadjoint problem. It turns out that not all "standard definitions" of the essential spectrum coincide in this example. We will then briefly discuss further results for generalisations of the model. This is joint work with Malcolm Brown (Cardiff), Tomas Dohnal (Halle), Karl Michael Schmidt (Cardiff) and Michael Plum (Karlsruhe).