THE GAUGE TRANSFORM

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1. Introduction

These notes were prepared for a course in Summer 2023 at the CRM on the Gauge transform and update for a course in Summer 2024 at the LMS Bath: Advances in Spectral Theory conference. The goal of the notes is to discuss the gauge transform and analysis in resonant zones as they are used to produce full asymptotics for periodic Schrodinger operators in dimension ≤ 2 with smooth potentials. The same ideas, albeit with more complicated notations and assumptions are used to produce the same result for a wide variety of almost periodic Schrödinger operators by Parnovski–Shterenberg. A more complicated version of the Gauge transform has been used to prove the full asymptotic expansion for all uniformly smoothly bounded potentials in one dimension in a recent paper by G–Parnovski–Shterenberg.

1.1. **Some basic notation.** Throughout these notes we use the following notation.

(1) $D_x := -i\partial_x$
(2) $\langle \xi \rangle := (1 + |\xi|^2)^{\frac{1}{2}}$.
(3) Let $\mathcal{B}$ be a Banach space and $f : (0,1) \to (0,\infty)$. We say that $u = O_\varepsilon(f(h))_{\mathcal{B}}$ if there are $h_0 > 0$ and $C > 0$ depending on the parameters ε such that

$$\|u\|_{\mathcal{B}} \leq Cf(h), \qquad 0 < h < h_0.$$

(4) Let $\mathcal{B}$ be a Banach space and $f : (0,1) \to (0,\infty)$. We say that $u = o(f(h))_{\mathcal{B}}$ if

$$\limsup_{h \to 0^+} \frac{\|u\|_{\mathcal{B}}}{f(h)} = 0.$$

(5) We say that $u = O_\varepsilon(h^\infty)_{\mathcal{B}}$ if there is $h_0 > 0$ depending on the parameters ε and for all $N > 0$ there is $C_N > 0$ depending on N and ε such that

$$\|u\|_{\mathcal{B}} \leq C_N h^N, \qquad 0 < h < h_0.$$

(6) $\mathbb{M}(m \times n)$ - the set of $m \times n$ matrices
(7) $\mathbb{S}(d \times d)$ - the set of $d \times d$ symmetric matrices.

2. The goal of these notes

Consider a Schrödinger operator

$$H := -\Delta + V,$$

where $V \in C^\infty(\mathbb{R}^d)$ such that for any $\alpha \in \mathbb{N}^d$,

$$\|\partial_x^\alpha V\|_{L^\infty} \leq C_\alpha. \tag{1}$$

When V satisfies (1), we write $V \in C_b^\infty(\mathbb{R}^d)$. In many ways, V is a small perturbation of $-\Delta$ and hence may have a small effect on the spectrum of $-\Delta$ at high energy. Crucially, however V need not decay fast toward $|x| = \infty$ and hence the perturbation is *not* relatively compact and may change the nature of the spectrum dramatically, even at high energy. Nevertheless, there are good reasons to think that these changes are in a sense small.

These notes will discuss one such manifestation of the 'smallness' of these changes. For this, we define the spectral projector for H onto the interval I as

$$E(H; I) := 1_I(H),$$

and, for $\lambda \in \mathbb{R}$, we define the *spectral function for H at λ* by

$$E(H)(\lambda, x, y) = E(H; (-\infty, \lambda])(x, y),$$

where $E(H; (-\infty, \lambda])(x, y)$ denotes the integral kernel of $E(H; (-\infty, \lambda])$. One can check easily using the fact that $-\Delta$ is elliptic and non-negative, that $E(H)(\lambda, x, y)$ is indeed a smooth function in (x, y). We will also sometimes write $E(H)(\lambda)$ as a shortened notation for $E(H; (-\infty, \lambda])$.

The *local density of states at x* is then given by $E(H)(\lambda, x, x)$. One manifestation of the 'smallness' of the perturbation V is contained in the next conjecture.

Conjecture 1. *Suppose that $V \in C_b^\infty(\mathbb{R}^d)$. Then, there are $a_j(x) \in C_b^\infty(\mathbb{R}^d)$ such that for any N and x, there is $C_{N,x}$ such that*

$$\Big|E(\lambda, x, x) - \sum_{j=0}^{N-1} \lambda^{\frac{d}{2}-j} a_j(x)\Big| \leq C_{N,x} \lambda^{\frac{d}{2}-N}.$$

As stated, the Conjecture 1 remains open, but it is known in a number of cases, for example.

(1) $d = 1$
(2) $d \geq 2$ and V almost periodic
(3) $d \geq 2$ and V decaying fast enough.
(4) $d = 2$ and V almost periodic plus decaying
(5) $d = 2$ and V radial

However, if one replaces V by a pseudodifferential operator of any positive order, this is false.

A slightly stronger version of this conjecture can be stated:

Conjecture 1'. Suppose that $V \in C_b^\infty(\mathbb{R}^d)$. Then, there are $a_j(x) \in C_b^\infty(\mathbb{R}^d)$ such that for any N there is C_N such that for all x

$$\Big|E(\lambda, x, x) - \sum_{j=0}^{N-1} \lambda^{\frac{d}{2}-j} a_j(x)\Big| \leq C_N \lambda^{\frac{d}{2}-N}.$$

In particular, the local density of states should have a complete asymptotic expansion in powers of λ, uniformly in x. This conjecture is false in any dimension higher than 1, but is true for subclasses of potentials:

(1) $d = 1$
(2) $d \geq 2$ and V almost periodic (+ some generic assumptions)

(3) $d = 2$ and V almost periodic.

If Conjecture 1' holds, then one can, of course directly obtain the full asymptotic expansion of the integrated density of states (when it exists) from a full asymptotic expansion for the local density of states.

Conjecture 2. *Suppose that $V \in C_b^\infty(\mathbb{R}^d)$. Then, there are $a_j(x) \in C_b^\infty(\mathbb{R}^d)$ such that for any N*

$$\limsup_{R\to\infty} \frac{1}{R^d} \int_{|x|\le R} \Big(E(\lambda, x, x) - \sum_{j=0}^{N-1} \lambda^{\frac{d}{2}-j} a_j(x) \Big) dx \le C_N \lambda^{\frac{d}{2}-N},$$

and

$$\liminf_{R\to\infty} \frac{1}{R^d} \int_{|x|\le R} \Big(E(\lambda, x, x) - \sum_{j=0}^{N-1} \lambda^{\frac{d}{2}-j} a_j(x) \Big) dx \ge -C_N \lambda^{\frac{d}{2}-N},$$

In particular, if the integrated density of states exists, then it has a full asymptotic expansion in powers of λ.

In these notes, we will discuss only the case of V periodic and $d \le 2$. In the case of dimension 1, we will, in fact, prove the full asymptotic expansion of the local density of states, while in dimension 2, for simplicity, we will only prove the full asymptotic expansion of the integrated density of states.

It will be convenient throughout these notes to make a semiclassical rescaling of the problem. That is, put $\lambda = \hbar^{-1}$ and consider the operator

$$H_\hbar = -\hbar^2 \Delta + \hbar^2 V,$$

and the spectral function

$$E(H_\hbar)(\omega, x, x)$$

for some $\omega > 0$. Observe that

$$E(H_\hbar)(\omega, x, x) = E(H, \hbar^{-2}\omega, x, x),$$

and hence we aim to prove that $E(H_\hbar)(\omega, x, x)$ (or its integrated version) has a complete asymptotic expansion in powers of $\hbar$. While this rescaling may seem unnatural, it will allow us to work in compact subsets of phase space and build the uniform estimates in the spectral parameter into our microlocal calculus.

From now on, we will actually drop the $_\hbar$ from our notation for the operator and, abusing notation somewhat, write $H = -\hbar^2 \Delta + \hbar^2 V$.

3. Basic semiclassical analysis

3.1. **Pseudodifferential operators on $\mathbb{R}^n$.** Pseudodifferential operators are quantizations of observables on the phase space, $T^*\mathbb{R}^n$, i.e. of functions $a = a(x, \xi) \in C^\infty(T^*\mathbb{R}^n)$ where we use coordinates (x, ξ) with $x \in \mathbb{R}^n$ and $\xi \in T_x^*\mathbb{R}^n$. In order to define pseudodifferential operators carefully, we first need to define symbol classes. In what follows, given $(x, \xi) \in T^*\mathbb{R}^n$ we write $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$.

Remark 3.1. In order to simplify notation, we typically allow functions and operators to implicitly depend on the small parameter h, but our constants are uniform in $0<h<1$.

Definition 3.2 (Symbol class)**.** We say that $a\in C^\infty(T^*\mathbb{R}^n)$ is a symbol of order $m\in\mathbb{R}$ and class $0\le\delta<\frac{1}{2}$ and write $a\in\tilde{S}^m_\delta(T^*\mathbb{R}^n)$ if for all $\alpha,\beta\in\mathbb{N}^n$, there is $C_{\alpha\beta}>0$ such that

$$|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|\le C_{\alpha\beta}h^{-\delta(|\alpha|+|\beta|)}\langle\xi\rangle^{m-|\beta|}.$$

We will actually use the slightly smaller class of symbols $S^m_\delta(T^*\mathbb{R}^n)$. Here, we will need an auxilliary parameter $\mu(h)$ with $ch\le\mu^{-1}$. We we say $a\in S^m_\delta(T^*\mathbb{R}^n)$ if there are $a_j\in\mu^{2\delta j}\tilde{S}^{m-j}_\delta$ depending on μ but not on h such that

$$\Big|\partial_x^\alpha\partial_\xi^\beta\Big(a-\sum_{j=0}^{N-1}h^ja_j\Big)\Big|\le C_{\alpha\beta}h^{N(1-2\delta)-\delta(|\alpha|+|\beta|)}.$$

We will often write simply $a\in S^m$ when the space is clear from context. We also define $S^\infty:=\bigcup_m S^m$, $S^{-\infty}:=\bigcap_m S^m$. We also define S^{comp} to be the set of $a\in S^{-\infty}$ which are supported in some h-independent compact set. Furthermore, we often write $S^m_0=S^m$, i.e., omit the $\delta=0$ in various spaces of symbol classes below.

It will also be be convenient to have a notion of semiclassical Sobolev spaces. In order to define these spaces, we first recall some standard definitions.

Definition 3.3 (Shwartz functions and distributions)**.** We define the space of Schwartz functions on $\mathbb{R}^n$ by

$$\mathscr{S}(\mathbb{R}^n):=\{u\in C^\infty(\mathbb{R}^n)\,:\,\sup_x|x|^\beta|\partial_x^\alpha u(x)|<C_{\alpha\beta},\ \text{ for all }\alpha,\beta\in\mathbb{N}^n\}.$$

The space of Schwartz distributions, $\mathscr{S}'(\mathbb{R}^n)$ is then the dual of $\mathscr{S}(\mathbb{R}^n)$.

We next recall the semiclassical Fourier transform.

Definition 3.4 (Semiclassical Fourier transform)**.** The semiclassical Fourier transform is the map $\mathcal{F}:\mathscr{S}'(\mathbb{R}^n)\to\mathscr{S}'(\mathbb{R}^n)$ given by

$$\mathcal{F}(u)(\xi):=\int_{\mathbb{R}^n}e^{\frac{i}{h}\langle x-y,\xi\rangle}u(y)dy.$$

We can now define the semiclassical Sobolev spaces. The elements of these spaces are the same as for the standard Sobolev spaces, but the norm is scaled in a way depending on h.

Definition 3.5 (Semiclassical Sobolev norm)**.** For $s\in\mathbb{R}$ the s-semiclassical Sobolev norm is defined as

$$H_h^s(\mathbb{R}^n):=\{u\in\mathscr{S}'(\mathbb{R}^n)\,:\,\langle\xi\rangle^s\mathcal{F}(u)\in L^2(\mathbb{R}^n)\},\qquad \|u\|^2_{H_h^s}:=(2\pi h)^{-n}\|\langle\xi\rangle^s\mathcal{F}(u)\|^2_{L^2}.$$

For $A:\mathscr{S}(\mathbb{R}^n)\to\mathscr{S}(\mathbb{R}^n)$, we say that $A=O(h^\infty)_{\Psi^{-\infty}}$ if for all N, there is $C_N>0$ such that

$$\|A\|_{H_h^{-N}\to H_h^N}\le C_Nh^N.$$

We may now introduce the class of pseudodifferential operators on $\mathbb{R}^n$.

Definition 3.6 (Pseudodifferential operator on $\mathbb{R}^n$)**.** For $m \in \mathbb{R}$, we say that A is a *pseudodifferential operator of order m* and write $A \in \Psi_\delta^m(\mathbb{R}^n)$ if there is $a \in S_\delta^m(T^*\mathbb{R}^n)$ such that

$$A = Op_h^W(a) + O(h^\infty)_{\Psi^{-\infty}}, \qquad [Op_h^W(a)u](x) := \frac{1}{(2\pi h)^d} \int e^{\frac{i}{h}\langle x-y,\xi\rangle} a\Big(\frac{x+y}{2}, \xi\Big) u(y) dy d\xi.$$

Here, the integral in $Op_h^W(a)u$, can be understood as an iterated integral when $u \in \mathscr{S}(\mathbb{R}^n)$ and it is not hard to check that operators in $\Psi_\delta^m(\mathbb{R}^n)$ are bounded on $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$.

Remark 3.7. The W in the notation Op_h^W stands for the Weyl quantization. There are many other standard choices of quantization including the left quantization. We refer the reader to [**?**, Chapter 4] for more information.

As with symbols, we sometimes omit the space $\mathbb{R}^n$ from the notation and define $\Psi_\delta^\infty := \bigcup_m \Psi_\delta^m$, $\Psi_\delta^{-\infty} := \bigcap_m \Psi_\delta^m$. We also define $\Psi_\delta^{\text{comp}}$ to be those $A \in \Psi_\delta^{-\infty}$ such that

$$A = Op_h^W(a) + O(h^\infty)_{\Psi^{-\infty}}$$

for some $a \in S_\delta^{\text{comp}}$. Furthermore, we sometimes write $\Psi_0^m = \Psi^m$ i.e. omit the $\delta = 0$ from spaces of operators below. In what follows we will need the following result of [**?**, Theorems 4.14,4.17] that explains the result of composition of two pseudodifferential operators.

3.2. **Symbol map.** We now recall the most important, basic properties of the pseudodifferential calculus [**?**, Appendix E].

Theorem 3.8 (Symbol map)**.** *There is a map*

$$\sigma_{m,\delta} : \Psi_\delta^m(\mathbb{R}^n) \to S_\delta^m(T^*\mathbb{R}^n)$$

such that the following holds.

1. *Suppose that $A \in \Psi_\delta^m$ and $\sigma_{m,\delta}(A) = 0$. Then $A \in h^{1-2\delta}\Psi_\delta^{m-1}$.*
2. *Suppose that $A \in \Psi_\delta^m$. Then, $A^* \in \Psi_\delta^m$ and $\sigma_{m,\delta}(A^*) = \overline{\sigma_{m,\delta}(A)}$.*
3. *Let $A \in \Psi_\delta^{m_1}$ and $B \in \Psi_\delta^{m_2}$. Then $AB \in \Psi_\delta^{m_1+m_2}$ and*
$$\sigma_{m_1+m_2,\delta}(AB) = \sigma_{m_1,\delta}(A)\sigma_{m_2,\delta}\sigma(B).$$
4. *Let $A \in \Psi_\delta^{m_1}$ and $B \in \Psi_\delta^{m_2}$. Then $[A,B] \in h^{1-2\delta}\Psi_\delta^{m_1+m_2-1}$ and*
$$\sigma_{m_1+m_2-1,\delta}(h^{2\delta-1}[A,B]) = -ih^{2\delta}\{\sigma_{m_1,\delta}(A), \sigma_{m_2,\delta}(B)\},$$
where $\{a,b\}$ denotes the Poisson bracket of a and b.

Remark 3.9. Usually, we will write σ for the symbol map, leaving the m, δ implicit.

Finally, we record the boundedness properties of pseudodifferential operators [**?**, Proof of Thoerem 13.13].

Lemma 3.10 (Boundedness properties)**.** *Let $A \in \Psi_\delta^m(\mathbb{R}^n)$. Then for any $s \in \mathbb{R}$ there is $C > 0$ such that for $0 < h < 1$,*

$$\|A\|_{H_h^s \to H_h^{s-m}} \le C.$$

3.3. **Wavefront set.** Before proceeding to properties of pseudodifferential operators such as ellipticity, we introduce the wavefront set of a pseudodifferential operator.

We can now define the essential support of a symbol and the wavefront set of a pseudodifferential operator. These notions codify the idea of the 'support' in phase space of a pseudodifferential operator.

Definition 3.11 (Essential support)**.** Let $a \in S^m_\delta$. For $(x_0,\xi_0) \in T^*\mathbb{R}^n$, we say that $(x_0,\xi_0) \notin \operatorname{ess\,supp}(a)$ if there is an h-independent neighborhood, U of (x_0,ξ_0) such that for all $\alpha,\beta \in \mathbb{N}^n$, and $N \in \mathbb{R}$, there is $C_{\alpha\beta N} > 0$ such that for $0 < h < 1$,

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha\beta N} h^N, \qquad (x,\xi) \in U.$$

Definition 3.12 (Wavefront set)**.** Let $A \in \Psi^m_\delta$. For $(x_0,\xi_0) \in T^*\mathbb{R}^n$, we say that $(x_0,\xi_0) \notin \mathrm{WF_h}(A)$ if there is $a \in S^m_\delta$ such that $(x_0,\xi_0) \notin \operatorname{ess\,supp}(a)$ and

$$A = Op_h(a) + O(h^\infty)_{\Psi^{-\infty}}.$$

It is easy to see from the definition that for any $A \in \Psi^m$, $\mathrm{WF_h}(A) \subset T^*\mathbb{R}^n$ is closed.

The crucial feature of the wavefront set is contained in the following lemma [**?**, (E.2.5)].

Lemma 3.13. *Suppose that* $A \in \Psi^{m_1}_\delta$, $B \in \Psi^{m_2}_\delta$. *Then,*

$$\mathrm{WF_h}(AB) \subset \mathrm{WF_h}(A) \cap \mathrm{WF_h}(B).$$

Remark 3.14. Let $A \in \Psi^m$. Because of Lemma 3.13, one may think of $\mathrm{WF_h}(A)$ as the set on which A 'lives'; i.e. Au contains no information about the parts of u which are not in $\mathrm{WF_h}(A)$. We will see in the next section (see Lemma 3.17) that when $|\sigma(A)(x,\xi)| > c\langle\xi\rangle^m$ for $(x,\xi) \in U \subset T^*M$, then Au encodes all the information about the function u on U.

Lemma 3.15. *Suppose that* $A \in \Psi^{\mathrm{comp}}_\delta$ *and* $\mathrm{WF_h}(A) = \emptyset$. *Then* $A = O(h^\infty)_{\Psi^{-\infty}}$.

The wavefront set, by definition, is an h-independent subset of $T\mathbb{R}^n$.

3.4. **Ellipticity and inverses.** We now define the notion of ellipticity for pseudodifferential operators.

Definition 3.16 (Ellipticity)**.** Let $A \in \Psi^m(M)$. For $(x_0,\xi_0) \in \overline{T}^*M$, we say that A *is elliptic at* (x_0,ξ_0), and write $(x_0,\xi_0) \in \mathrm{Ell}(A)$, if there is a neighborhood, $U \subset \overline{T}^*M$ of (x_0,ξ_0) and $c > 0$ such that

$$|\sigma(A)(x,\xi)| \ge c\langle\xi\rangle^m, \qquad (x,\xi) \in U \cap T^*M$$

It is easy to see from the definition that for any $A \in \Psi^m(M)$, $\mathrm{Ell}(A) \subset \overline{T}^*M$ is open.

Ellipticity gives an appropriate conditions which guarantee that A is invertible on a subset of $\overline{T}^*M$ in the following sense.

Lemma 3.17 (Elliptic parametrix)**.** *Suppose that* $A \in \Psi^{m_1}_\delta$ *and* $B \in \Psi^{m_2}_\delta$ *with* $\mathrm{WF_h}(B) \subset \mathrm{Ell}(A)$. *Then there are* $E_L, E_R \in \Psi^{m_2-m_1}_\delta$ *such that*

$$B = E_L A + O(h^\infty)_{\Psi^{-\infty}}, \qquad B = AE_R + O(h^\infty)_{\Psi^{-\infty}}.$$

As with many constructions in semiclassical analysis, this lemma is proved by an iterative construction. The nonlinear part of the construction is done by solving a top order equation, and then each successive iteration involves only the solution of a linear equation. In the case of the elliptic parametrix construction, this is particularly simple since the equations involved are algebraic.

Proof. Let $e = \sigma(B)/\sigma(A)$. Then, since $\mathrm{WF_h}(B) \subset \mathrm{Ell}(A)$, $|\sigma(A)| > c > 0$ on $\operatorname{supp}\sigma(B)$, and hence, $e \in S_\delta^{m_2-m_1}$. Putting $E_{L,0} := Op_h(e)$, we have

$$\sigma_{m_2,\delta}(E_{L,0}A - B) = 0,$$

and therefore,

$$E_{L,0}A = B + h^{1-2\delta}R_1,$$

with $R_1 \in \Psi_\delta^{m_1-1}$.

Suppose we have found e_j, $i = 0, 1, \ldots, N-1$, $e_j \in S_\delta^{m_2-m_1-j}$ such that $\operatorname{supp} e_j \subset \mathrm{WF_h}(B)$, and, with $E_{L,N-1} := \sum_{j=0}^{N-1} h^{j(1-2\delta)}Op_h(e_j)$, we have

$$E_{L,N-1}A = B + h^{N(1-2\delta)}R_N, \tag{2}$$

for some $R_N \in \Psi_\delta^{m_2-N}$. Now, since $\operatorname{supp} e_i \subset \mathrm{WF_h}(B)$, $\mathrm{WF_h}(E_{L,N-1}) \subset \mathrm{WF_h}(B)$ and hence,

$$\mathrm{WF_h}(R_N) = \mathrm{WF_h}(h^{-N(1-2\delta)}(B - E_{L,N-1}A)) \subset \mathrm{WF_h}(B).$$

Therefore $\mathrm{WF_h}(R_N) \subset \mathrm{Ell}(A)$ and hence $e_N := -\sigma(R_N)/\sigma(A) \in S_\delta^{m_2-N-m_1}$ and

$$(E_{L,N-1} + h^{N(1-2\delta)}Op_h(e_N))A - B = h^{N(1-2\delta)}(R_N + Op_h(e_N)A) \in h^{N(1-2\delta)}\Psi^{m_2-N},$$

and

$$\sigma_{m_2-N,\delta}(R_N + Op_h(e_N)A) = 0.$$

Therefore,

$$(E_{L,N-1} + h^{N(1-2\delta)}Op_h(e_N))A - B = h^{(N+1)(1-2\delta)}R_{N+1},$$

for some $R_{N+1} \in \Psi^{m_2-N-1}$. In particular, putting $E_{L,N} = \sum_{j=0}^{N} h^j Op_h(e_j)$, we have (2) with $N-1$ replaced by N. In particular, there are $e_j \in \Psi_\delta^{m_2-m_2-j}$ for $j = 0, 1, \ldots$ such that (2) holds for any N. Setting $E_L \sim \sum_j h^{j(1-2\delta)}Op_h(e_j)$, completes the proof of the first equality.

The proof of the second equality is nearly identical and we leave the details to the reader. □

3.5. **Auxilliary facts.**

Lemma 3.18. *Suppose that $P \in \Psi^m$ is self-adjoint such that*

$$|\sigma(P)(x,\xi)| \geq c|\xi|^m - C, \quad (x,\xi) \in T^*\mathbb{R}^n.$$

Then, for all $\chi \in C_c^\infty$, $\chi(P) \in \Psi^{\mathrm{comp}}$ and

$$\mathrm{WF_h}(\chi(P)) \subset \{(x,\xi) : p(x,\xi) \in \operatorname{supp}\chi\}, \qquad \mathrm{WF_h}(I - \chi(P)) \subset \{(x,\xi) : p(x,\xi) \in \operatorname{supp}(1-\chi)\}.$$

Proof. TODO? □

Lemma 3.19. *Let $A \in \Psi^0$. Then $e^{iA} \in \Psi^0$.*

Proof. We will construct an asymptotic series $b_j \in S^{-j}$ such that, with

$$B \sim \sum_j \hbar^j b_j,$$

we have $e^{itA} = Op_h(Be^{it\sigma(A)}) + O(\hbar^\infty)_{\Psi^{-\infty}}$. Set $b_0 = 1$. Then,

$$\begin{aligned} D_t(e^{-itA}Op_h(b_0e^{it\sigma(A)}) &= e^{-itA}(-AOp_h(e^{it\sigma(A)}) + Op_h(\sigma(A)e^{it\sigma(A)})) \\ &= e^{-itA}\hbar R_1(t)Op_h(e^{it\sigma(A)}) \end{aligned}$$

for some $R_1(t) \in \Psi^{-1}$. Now, suppose that we have $b_j(t) \in S^{-j}$ such that, with $B_{N-1}(t) := \sum_{j=0}^{N-1} \hbar^j b_j(t)$, we have

$$D_t(e^{-itA}Op_h(B_{N-1}(t)e^{it\sigma(A)})) = e^{-itA}\hbar^N R_N(t)Op_h(e^{it\sigma(A)}),$$

for some $R_N \in \Psi^{-N}$. Then, let $b_N(t)$ solve

$$b_N'(t) - i\sigma(A)b_N(t) = -\sigma(R_N), \qquad b_N(0) = 0,$$

i.e.

$$b_N = -\int_0^t e^{i(t-s)\sigma(A)}\sigma(R_N)(s)ds \in \Psi^{-N}.$$

We have

$$\begin{aligned} &D_t(e^{-itA}Op_h([B_{N-1}(t) + h^N b_N(t)]e^{it\sigma(A)})) \\ &= e^{-itA}h^N R_N Op_h(e^{it\sigma(A)}) + h^N e^{-itA}(Op_h((-ib_N(t)\sigma(A) + b_N'(t))e^{tit\sigma(A)})) \\ &= e^{-itA}h^N R_N Op_h(e^{it\sigma(A)}) + h^N e^{-itA}(Op_h((-ib_N(t)\sigma(A) + b_N'(t)) + hR_{N+1})Op_h(e^{tit\sigma(A)})) \\ &= h^{N+1}e^{-itA}R_{N+1}Op_h(e^{it\sigma(A)}), \end{aligned}$$

where $R_{N+1} \in \Psi^{-N-1}$. Then, by induction, setting $B(t) \sim \sum_j h^j b_j(t)$, we have

$$D_t(e^{-itA}Op_h(B(t)e^{it\sigma(A)})) = e^{itA}O(\hbar^\infty)_{\Psi^{-\infty}}.$$

So that

$$e^{-iA}Op_h(B(1)e^{i\sigma(A)}) = I + O(\hbar^\infty)_{H_\hbar^{-s}\to L^2}.$$

Hence,

$$e^{itA} - Op_h(B(t)e^{it\sigma(A)}) = O(\hbar^\infty)_{H_\hbar^{-s}\to L^2}$$

and, in particular $e^{itA} : H_\hbar^{-s} \to H_\hbar^{-s}$ is bounded uniformly in h.

Repeating the construction, we also find $\tilde{B}(t)$ such that

$$D_t(Op_h(\tilde{B}(t)e^{it\sigma(A)})e^{-itA}) = O(\hbar^\infty)_{\Psi^{-\infty}}e^{-itA},$$

and hence, since $e^{itA} : H_\hbar^{-s} \to H_\hbar^{-s}$ is uniformly bounded,

$$O(\hbar^\infty)_{\Psi^{-\infty}}e^{-itA} = O(\hbar^\infty)_{\Psi^{-\infty}}.$$

In particular,

$$e^{itA} - Op_h(\tilde{B}(t)e^{it\sigma(A)}) = O(\hbar^\infty)_{\Psi^{-\infty}}.$$

□

Lemma 3.20. *For $B \in \Psi^m_\delta$, $A_\delta \in \hbar^\ell \Psi^0$, and any N*

$$e^{iA}Be^{-iA} - \sum_{j=0}^{N-1} \frac{i^j}{j!} \operatorname{ad}_A^j B \in \hbar^{N(1-2\delta+\ell)}\Psi_\delta^{m-N}.$$

In particular,

$$e^{iA}Be^{-iA} - B - i[A,B] \in \hbar^{2(1-2\delta+\ell)}\Psi_\delta^{m-N}.$$

Proof. Observe that

$$D_t^k e^{itA}Be^{-itA} = e^{itA}\operatorname{ad}_A^k Be^{-itA}.$$

Thus,

$$e^{iA}Be^{-iA} = \sum_{j=0}^{N-1} \frac{i^j}{j!}\operatorname{ad}_A^j B + \int_0^1 \frac{i^N}{(N-1)!}(1-t)^{N-1}e^{itA}\operatorname{ad}_A^N Be^{-itA}dt.$$

Now, $e^{itA} \in \Psi^0$, and $\operatorname{ad}_A^N B \in h^{N(1-2\delta+\ell)}\Psi_\delta^{m-N}$. Therefore,

$$\int_0^1 \frac{i^N}{(N-1)!}(1-t)^{N-1}e^{itA}\operatorname{ad}_A^N Be^{-itA}dt \in \hbar^{N(1+\ell)}\Psi^{m-N},$$

and hence the lemma follows.

Remark 3.21. Really all we need here is that $e^{iA} : H^s_\hbar \to H^s_\hbar$ for any s.

$$e^{iA}B - Be^{iA} = (e^{iA}Be^{-iA} - B)e^{iA}$$

□

4. Basic reductions

We start by discussing the natural requirements for the spectral function of two operators to be close. First, notice that closeness of two operators, H_1 and H_2 in *any* norm does not suffice for the spectral projectors, $E(H_j)(\lambda)$ to be close to each other. For example, consider

$$H_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2+\varepsilon \end{pmatrix},$$

Then,

$$E(H_1)(2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad E(H_2)(2) = \begin{pmatrix} 1 & 0 \\ 0 & 1_{(-\infty,0]}(\varepsilon) \end{pmatrix}.$$

Indeed, an eigenvalue of H_1 may be perturbed out of $(-\infty,\lambda]$ and hence, a small perturbation may cause a large change in the spectral projector *in any topology*. If, however, we consider $E(H_1)(\lambda)$ for $\lambda \notin \operatorname{Spec}(H_1)$ the spectral projectors will always be close. This assumption is too much for our purposes since we typically expect the spectrum of our operators to include most of $[0,\infty)$ e.g. for periodic Schrödinger operators. Therefore, we need something a little weaker.

Notice that

$$E(H)(\lambda,x,y) = \langle E(H)(\lambda)\delta_x, \delta_y\rangle = \langle E(H)(\lambda)\delta_x, E(H)(\lambda)\delta_y\rangle,$$

so we can work in the strong topology.

In particular, an important ingredient in the proof is the smallness of

$$E(H_2;(\lambda-\iota,\lambda+\iota])\delta_x = E(H_2)(\lambda+\iota,x,x) - E(H_2)(\lambda-\iota,x,x) \tag{3}$$

for small ι.

Our next Lemma will be used to show that if two operators are close near a particular energy level, then their spectral projectors are close in the strong topology near that energy level (see Lemma 4.2). First, we prove the following lemma.

Lemma 4.1. *Let $\mathcal{H}$ be a Hilbert space, $a\in\mathbb{R}$, $s\geq 0$, $J\subset\mathbb{R}$ an interval and H_1,H_2 be self-adjoint operators on $\mathcal{H}$ with $H_j\geq a$ for $j=1,2$. Define $J_-:=J^c\cap(-\infty,\inf J]$ and $J_+:=J^c\cap[\sup J,\infty)$, and*

$$\begin{aligned}\varepsilon_1 &:= \|E(H_1;J_-)(H_1-H_2)E(H_2;J_+)(H_2+(1-a)I)^s\|,\\ \varepsilon_2 &:= \|(H_1-H_2)E(H_2;J)(H_2+(1-a)I)^s\|,\\ \varepsilon_3 &:= \|E(H_1;J)(H_1-H_2)(H_2+(1-a)I)^s\|.\end{aligned} \tag{4}$$

Suppose that $\lambda-a\geq 1$ and $[\lambda-\iota,\lambda+\iota]\subset J$. Then,

$$\|E(H_1;(-\infty,\lambda-\iota])E(H_2;[\lambda+\iota;\infty))(H_2-a+1)^s\| \leq \frac{\pi(\varepsilon_1+\varepsilon_2+\varepsilon_3)}{\iota}.$$

Proof. Assume that

$$\phi = E(H_1;(-\infty,\lambda-\iota])\phi, \qquad (H_2-a+1)^s\psi = E(H_2;[\lambda+\iota,\infty))(H_2-a+1)^s\psi, \tag{5}$$

with $\|\phi\|=\|\psi\|=1$. Then we need to establish $|(\phi,(H_2-a+1)^s\psi)|\leq \frac{\pi(\varepsilon_1+\varepsilon_2+\varepsilon_3)}{\iota}$. We have

$$\begin{aligned}(\phi,(H_s-a+1)^s\psi) &= \int_\gamma \langle (H_1-z)^{-1}\phi,(H_2-a+1)^s\psi\rangle dz\\ &= \int_\gamma \langle \phi,(H_1-\bar z)^{-1}(H_2-a+1)^s\psi\rangle dz\\ &= \int_\gamma \langle \phi,(H_2-\bar z)^{-1}+(H_1-\bar z)^{-1}(H_1-H_2)(H_2-\bar z)^{-1}(H_2-a+1)^s\psi\rangle dz\\ &= \int_\gamma \langle \phi,(H_1-\bar z)^{-1}(H_1-H_2)(H_2-\bar z)^{-1}(H_2-a+1)^s\psi\rangle dz\\ &= \int_\gamma ((H_1-z)^{-1}\phi,(H_1-H_2)(H_2-a+1)^s(H_2-\bar z)^{-1}\psi\rangle dz\end{aligned}$$

where $\gamma=\gamma_N$ is the closed square contour in the complex plane symmetric about $\mathbb{R}$ and intersecting $\mathbb{R}$ at λ and $-N$ where $N>-a$ is large. Note that in the next to last line we have used that with $\bar\gamma$ the contour conjugate to γ,

$$\int_{\bar\gamma}(H_2-\bar z)^{-1}E((\lambda+\iota,\infty];H_2)d\bar z = 0.$$

Now,

$$\begin{aligned}
&(H_1 - H_2)(H_2 - a + 1)^s \\
&\quad = E(H_1; J)(H_1 - H_2)(H_2 + (1-a)I)^s + E(H_1; J^c)(H_1 - H_2)E(H_2; J^c)(H_2 + (1-a)I)^s \\
&\qquad + E(H_1; J^c)(H_1 - H_2)E(H_2; J)(H_2 + (1-a)I)^s.
\end{aligned}$$

Therefore, we need only to estimate the three terms

$$\begin{aligned}
I &:= \Big|\int_\gamma ((H_1 - z)^{-1}\phi, E(H_1; J^c)(H_1 - H_2)E(H_2; J^c)(H_2 - a + 1)^s (H_2 - \bar{z})^{-1}\psi) dz\Big|, \\
II &:= \Big|\int_\gamma ((H_1 - z)^{-1}\phi, E(H_1; J)(H_1 - H_2)(H_2 - a + 1)^s (H_2 - \bar{z})^{-1}\psi) dz\Big|, \\
III &:= \Big|\int_\gamma ((H_1 - z)^{-1}\phi, E(H_1; J^c)(H_1 - H_2)E(H_2; J)(H_2 - a + 1)^s (H_2 - \bar{z})^{-1}\psi) dz\Big|.
\end{aligned}$$

For I, we observe using (5) that

$$\begin{aligned}
\lim_{N\to\infty} I &= \lim_{N\to\infty} \Big|\int_\gamma ((H_1 - z)^{-1}\phi, E(H_1; J_-)(H_1 - H_2)E(H_2; J_+)(H_2 - a + 1)^s (H_2 - \bar{z})^{-1}\psi) dz\Big| \\
&\le \varepsilon_1 \lim_{N\to\infty} \Big(\int_\gamma \|(H_1 - z)^{-1}\phi\|^2 |dz|\Big)^{1/2} \Big(\int_\gamma \|(H_2 - z)^{-1}\psi\|^2 |dz|\Big)^{1/2} \le \frac{\pi\varepsilon_1}{\iota}.
\end{aligned}$$

Similarly, we estimate

$$\lim_{N\to\infty} II + III \le \frac{\pi(\varepsilon_2 + \varepsilon_3)}{\iota}$$

to finish the proof. □

We then use Lemma 4.1 to estimate the difference between spectral projectors in the strong operator topology.

Lemma 4.2. *Let $\mathcal{H}$ be a Hilbert space, $a \in \mathbb{R}$, $s \ge 0$, and H_1, H_2 be self-adjoint operators on $\mathcal{H}$ with $H_j \ge a$ for $j = 1, 2$. Define $\varepsilon_1, \varepsilon_2, \varepsilon_3$ as in (4). Then, if $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 < 1$, and $[\lambda - \iota, \lambda + \iota] \subset J$, for any $f \in \mathcal{H}$, $\lambda \ge a + 1$, and $\iota > 0$,*

$$\begin{aligned}
\|[E(H_1)(\sqrt{\lambda}) - E(H_2)(\sqrt{\lambda})]f\|_{\mathcal{H}} &\le 2\|E(H_2; [\lambda - \iota, \lambda + \iota])f\|_{\mathcal{H}} \\
&\quad + \frac{2\pi(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)}{\iota}\Big(\|E(H_2)(\sqrt{\lambda})f\|_{\mathcal{H}} + \|(H_2 + (1-a)I)^{-s} f\|_{\mathcal{H}}\Big).
\end{aligned} \tag{6}$$

Proof. Consider

$$\begin{aligned}
E(H_2)(\sqrt{\lambda})f &= E(H_2; (-\infty, \lambda - \iota))f + E(H_2; [\lambda - \iota, \lambda])f \\
&= [E(H_1; (-\infty, \lambda]) + E(H_1; (\lambda, \infty))]E(H_2; (-\infty, \lambda - \iota))f + E(H_2; [\lambda - \iota, \lambda])f.
\end{aligned}$$

Now, using Lemma 4.1 with $s = 0$, λ replaced by $\lambda + \frac{\iota}{2}$ and ι replace by $\frac{\iota}{2}$, we have

$$\begin{aligned}
\|E(H_1; (\lambda, \infty))E(H_2; (-\infty, \lambda - \iota))f\| &= \|E(H_1; (\lambda, \infty))E(H_2; (-\infty, \lambda - \iota))E(H_2)(\sqrt{\lambda})f\| \\
&\le 2\pi \frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_3}{\iota} \|E(H_2)(\sqrt{\lambda})f\|.
\end{aligned}$$

Now,

$$E(H_1;(-\infty,\lambda])E(H_2;(-\infty,\lambda-\iota))f = E(H_1)(\sqrt{\lambda})f - E(H_1;(-\infty,\lambda]))E(H_2;[\lambda-\iota,\infty))f,$$

and, using Lemma 4.1 again on the second term

$$\begin{aligned}&\|E(H_1;(-\infty,\lambda]))E(H_2;[\lambda-\iota,\infty))f\| \\ &\le \|E(H_1;(-\infty,\lambda]))E(H_2;[\lambda-\iota,\lambda+\iota))f\| \\ &\qquad + \|E(H_1;(-\infty,\lambda]))E(H_2;[\lambda+\iota,\infty))(H_2-a+1)^s(H_2-a+1)^{-s}f\| \\ &\le \|E(H_2;[\lambda-\iota,\lambda+\iota))f\| + \frac{2\pi(\varepsilon_1+\varepsilon_2+\varepsilon_2)}{\iota}\|(H_2-a+1)^{-s}f\|,\end{aligned}$$

which completes the proof. ☐

Corollary 4.3. *Let $\mathcal{H}$ be a Hilbert space, $a\in\mathbb{R}$, $s\ge 0$, and H_1, H_2 be self-adjoint operators on $\mathcal{H}$ with $H_j\ge a$ for $j=1,2$. Define*

$$\varepsilon := \|(H_1-H_2)(H_2+(1-a)I)^s\|, \tag{7}$$

Suppose that $\lambda - a\ge 1$. Then, Then, if $\varepsilon<1$ for any $f\in\mathcal{H}$, $\lambda\ge a+1$, and $\iota>0$,

$$\begin{aligned}\|[E(H_1)(\sqrt{\lambda})-E(H_2)(\sqrt{\lambda})]f\|_{\mathcal{H}} &\le 2\|E(H_2;[\lambda-\iota,\lambda+\iota])f\|_{\mathcal{H}} \\ &\quad + \frac{3\pi\varepsilon}{\iota}\Big(\|E(H_2)(\sqrt{\lambda})f\|_{\mathcal{H}} + \|(H_2+(1-a)I)^{-s}f\|_{\mathcal{H}}\Big).\end{aligned} \tag{8}$$

Proof. Observe that $\varepsilon_1,\varepsilon_2,\varepsilon_3$ in (4) all satisfy $\varepsilon_i\le\varepsilon$. ☐

4.1. Application to Schrödinger operators.

Lemma 4.4. *Let $\delta>0$, $V\in\Psi^{2-\delta}$, and put*

$$H := -\hbar^2\Delta + \hbar^\delta V.$$

Then, for any $\omega\in\mathbb{R}$,

$$\|E(H)(\omega)f\|_{L^2}\le C\langle\omega\rangle^s\|f\|_{H_\hbar^{-s}}.$$

Proof. Observe that $H\ge -1$ for $\hbar$ small enough. Therefore,

$$\begin{aligned}\|E(H)(\omega)f\|_{L^2} &= \|E(H)(\omega)(H+1)^s(H+1)^{-s}f\|_{L^2} \\ &\le (\omega^2+1)^s\|E(H)(\omega)(H+1)^{-s}f\|_{L^2}\le(\omega^2+1)^s\|(H+1)^{-s}f\|_{L^2}.\end{aligned}$$

Next, since $(H+1)^{-s}\in\Psi^{-2s}$, the lemma follows. ☐

Lemma 4.5. *Let $\delta_1,\delta_2>0$, $V_1,V_2\in\Psi^{2-\delta_1}$, and put*

$$H_1 := -\hbar^2\Delta+\hbar^{\delta_2}V_1, \qquad H_2 := -\hbar^2\Delta+\hbar^{\delta_2}V_2.$$

Let $0<\varepsilon<\min(\frac12,\delta_2)$ and $a(\hbar)<b(\hbar)-5\hbar^\varepsilon$ and $V_1(x,\xi)=V_2(x,\xi)$ for $a\le|\xi|^2\le b$. Then, with $J=[a+\hbar^\varepsilon, b-\hbar^\varepsilon]$, for any s, we have

$$\begin{aligned}\|E(H_1;J_-)(H_1-H_2)E(H_2;J_+)(H_2+1)^s\| &= O(\hbar^\infty), \\ \|(H_1-H_2)E(H_2;J)(H_2+1)^s\| &= O(\hbar^\infty), \\ \|E(H_1;J)(H_1-H_2)(H_2+1)^s\| &= O(\hbar^\infty).\end{aligned} \tag{9}$$

Proof. Observe that $J_- = a + \hbar^\varepsilon$, $J_+ = b - \hbar^\varepsilon$. Let $\chi_\pm \in S_\varepsilon$ with $\chi_- \equiv 1$ on $(-\infty, J_- + \hbar^\varepsilon)$, $\operatorname{supp}\chi_- \subset (-\infty, b - 2\hbar^\varepsilon)$, and $\chi_+ \equiv 1$ on $(J_+ - \hbar^\varepsilon, \infty)$ with $\operatorname{supp}\chi_+ \subset (a + 2\hbar^\varepsilon, \infty)$. Then,

$$\begin{aligned} &E(H_1; J_-)(H_1 - H_2)E(H_2; J_+)(H_2 + 1)^s \\ &\qquad = E(H_1; J_-)\chi_-(H_1)h^{\delta_2}(V_1 - V_2)\chi_+(H_2)E(H_2; J_+)(H_2 + 1)^s = O(\hbar^\infty)_{\Psi^{-\infty}} \end{aligned}$$

since $\chi_-(H_1), \chi_+(H_2) \in \Psi^0_\varepsilon$ with $\mathrm{MS_h}(\chi_-(H_1)) \cap \mathrm{MS_h}(\chi_+(H_2)) = \emptyset$.

Remark 4.6. If $\varepsilon = 0$, use wavefront set.

Now, let $\chi \in S_\varepsilon$ with $\chi \equiv 1$ on $[a + \hbar^\varepsilon, b - \hbar^\varepsilon]$ with $\operatorname{supp}\chi \subset [a + \frac{1}{2}\hbar^\varepsilon, b - \frac{1}{2}\hbar^\varepsilon]$. Then, $\chi(H_1), \chi(H_2) \in \Psi_\varepsilon$ and $\mathrm{MS_h}(\chi(H_i)) \subset \{a \le |\xi|^2 \le b\}$, hence

$$(H_1 - H_2)E(H_2; J) = \hbar^{\delta_2}(V_1 - V_2)\chi(H_2)E(H_2; J) = O(\hbar^\infty)_{\Psi^{-\infty}},$$

and

$$E(H_1; J)(H_1 - H_2) = E(H_1; J)\chi(H_1)\hbar^{\delta_2}(V_1 - V_2) = O(\hbar^\infty)_{\Psi^{-\infty}}.$$

□

Corollary 4.7. *Let $\delta_1, \delta_2 > 0$, $V_1, V_2 \in \Psi^{2-\delta_1}$, and put*

$$H_1 := -\hbar^2\Delta + \hbar^{\delta_2}V_1, \qquad H_2 := -\hbar^2\Delta + \hbar^{\delta_2}V_2.$$

Let $0 < \varepsilon < \min(\frac{1}{2}, \delta_2)$ and $a(\hbar) < b(\hbar) - 5\hbar^\varepsilon$ and $V_1(x,\xi) = V_2(x,\xi)$ for $a \le |\xi|^2 \le b$. Then, for $\omega^2 \in [a + \hbar^\varepsilon + \iota, b - \hbar^\varepsilon - \iota$, we have

$$\|E(H_1)(\omega)f - E(H_2)(\omega)f\| \le \|E(H_2; [\lambda - \iota, \lambda + \iota])f\|_{L^2} + O(\hbar^\infty\iota^{-1})\|f\|_{H_\hbar^{-s}}$$

Proof. The corollary follows from combining Lemma 4.2 with Lemmas 4.4 and 4.5. □

Remark 4.8. Given V_1, we can replace V_1 by V_2 which agrees with V_1 near $|\xi|^2 \in [a, b]$ and is zero outside a small neighborhood thereof provided that we have

$$\|E(H_2; [\lambda - \iota, \lambda + \iota])f\|_{L^2} \ll 1.$$

For the spectral function, observe that

$$E(H_1)(\sqrt{\lambda}, x, x) = \langle E(H_1)(\lambda)\delta_x, E(H_1)(\lambda)\delta_x\rangle.$$

Hence, it is enough to have that

$$\|E(H_2; [\lambda - \iota, \lambda + \iota])f\|_{L^2} = \langle E(H_2; [\lambda - \iota, \lambda + \iota])\delta_x, \delta_x\rangle \le E(H_2)(\sqrt{\lambda + 2\iota})(x,x) - E(H_2)(\sqrt{\lambda - 2\iota})(x,x) \ll 1.$$

To see that the spectral function for H_2 and H_1 are close. For instance, it is enough to have a complete asymptotic expansion for the spectral function of H_2. The upshot of all of this so far is that we can work with an operator which is *equal* to the Laplacian away from some particular energy surface.

5. Gauge transforms

Let $\chi \in C_c^\infty((-\frac{1}{2}, \frac{1}{2}))$ with $\chi \equiv 1$ near 0. We now start with the operator

$$H_2 := -\hbar^2\Delta + \hbar^2\chi(|\hbar D| - 1)V\chi(|\hbar D| - 1).$$

Our goal is to construct a unitary operator, U such that

$$UH_2U^* = H_3 + O(\hbar^\infty)_{H_\hbar^{-s}\to H_\hbar^s}, \qquad H_3 := -\hbar^2\Delta + \hbar^2 m(\hbar D) + \hbar^2 X_{res}$$

where X_{res} is sufficiently structured so that one can compute the spectral function for H_3. Observe that, using Corollary 4.3, it will be enough to understand the spectral projector of $\tilde{H}_3 := U^*H_3U$, (and hence that of H_3) provided that we have

$$E(\tilde{H}_3)(\sqrt{\lambda + 2\iota}, x, x) - E(\tilde{H}_3)(\sqrt{\lambda - 2\iota}, x, x) \ll 1$$

In particular, if we can show that $E(\tilde{H}_3)(\rho)(x, x)$ has a full asymptotic expansion in powers of ρ, then we will obtain the same for H_2 and hence H_1.

Ideally, we would like to have $X_{res} = 0$ since the spectral function of a Fourier multiplier is easy to compute. However, we will see that it is not possible to do this in dimension larger than 1.

Lemma 5.1. *Let $m \in C^\infty(\mathbb{R}^d)$. Then,*

$$E(m(\hbar D); (-\infty, \omega])(x, y) = \frac{1}{(2\pi\hbar)^d}\int_{m(\xi)\le\omega} e^{\frac{i}{\hbar}\langle x-y,\xi\rangle}d\xi.$$

Proof. Since the semiclassical Fourier transform diagonalizes $m(\hbar D)$; i.e. $\mathcal{F}_h m(\hbar D)\mathcal{F}_h^{-1} = m(\xi)$, we have

$$E(m(\xi); (-\infty, \omega]) = 1_{m(\xi)\le\omega}$$

Therefore

$$E(m(\hbar D); (-\infty, \omega]) = \mathcal{F}_h^{-1}1_{m(\xi)\le\omega}\mathcal{F}_h,$$

and the claim follows. □

Remark 5.2. In fact, it is possible to compute the spectral function for operators more general than Fourier multiplier. e.g. $-\hbar^2\Delta + \hbar^2 V$, where $V \in C_c^\infty$. We, however, will not discuss this here.

5.1. Basic Examples of Gauges transforms.

5.1.1. *An operator on $\mathbb{S}^1$.* Consider first

$$H := hD_{x_1} + hV : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1).$$

Our goal is to compute e.g. the spectrum of H and, to this end, we want to 'remove' V via a unitary conjugation. In this case, we consider

$$H_1 := e^{if}He^{-if} = hD_{x_1} - hf' + hV.$$

Ideally, we would like to solve $f' = V$, but this may not be possible since f should be a periodic function and hence

$$\int_0^{2\pi} f'(x)dx = 0.$$

We can, however, set

$$f(x) = \int_0^x V(t) - \bar{V} dt, \qquad \bar{V} := \frac{1}{2\pi}\int_0^2 \pi V(s)ds.$$

so that

$$H_1 = hD_x + h\bar{V}.$$

Notice that $\bar{V}$ is a constant and, in particular, commutes with hD_x. Hence it is easy to see that

$$\sigma(H_1) = \sigma(H) = \{n + h\bar{V} \,:\, n \in \mathbb{Z}\}.$$

5.1.2. *A 2×2 matrix.* Consider the matrix

$$H := \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 0 \end{pmatrix}$$

It is, of course, possible to diagonalize the matrix explicitly, but we want to emulate the procedure below. Hence, assume $A = O(\varepsilon)$ and write

$$e^{iA}He^{-iA} = H + i[A,H] + O(\varepsilon^2)$$

Hence, we solve

$$\begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix} = -i[A,H] + O(\varepsilon^2) = -i[A,H_0] + O(\varepsilon^2), \qquad H_0 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Doing this, we arrive at

$$A := \begin{pmatrix} 0 & -i\varepsilon \\ i\varepsilon & 0 \end{pmatrix},$$

and

$$e^{iA}He^{-iA} = \begin{pmatrix} 1 + O(\varepsilon^2) & O(\varepsilon^2) \\ O(\varepsilon^2) & O(\varepsilon^2) \end{pmatrix}.$$

We can then repeat the procedure to diagonalize modulo $O(\varepsilon^\infty)$.

5.2. **Discussion around the Gauge transform.** Consider

$$D_t(e^{itA}H_2e^{-itA}) = e^{itA}[A,H_2]e^{-itA},$$

so that

$$e^{iA}H_2e^{-iA} = i\int_0^1 [A, e^{itA}H_2e^{-itA}]dt + H_2$$

Since we want $e^{iA}H_2e^{-iA} = H_2 - \hbar^2 V + \hbar^2 B$, where B is pseudodiffferential, it is natural to look for $[A,H_2] = O(\hbar^2)_{\Psi^{\mathrm{comp}}}$, and hence $A \in \hbar\Psi^{\mathrm{comp}}$. Thus, we are in the situation of Lemma 3.20 and we have

$$e^{iA}H_2e^{-iA} - H_2 + i[A,H_2] \in \hbar^4\Psi^{\mathrm{comp}}$$

Now,

$$[A,H_2] - [A, -\hbar^2\Delta] \in \hbar^4\Psi^{\mathrm{comp}},$$

So, we aim to find

$$i\hbar^{-2}[A,\hbar^2\Delta] = V + l.o.t.$$

In particular, $A = \hbar Op_h(a)$, where

$$-H_{|\xi|^2}a = -2\langle \xi, \partial_x\rangle a = V.$$

So the question is: when is there a symbolic solution to this equation. Obvious problem– growth as $x\to\infty$ e.g. $V=1$.

Next, question: What is 'safe' to leave behind? Obvious answer: Fourier multipliers – spectral function for Fourier multipliers.

So, we aim to solve

$$-H_{|\xi|^2}a = -2\langle \xi, \partial_x\rangle a = V + S,$$

where S is safe. In the case of periodic operators, this is more natural on the Fourier transform side

$$-2i\langle \xi, \theta\rangle \hat{a}(\xi,\theta) = \hat{V}(\theta), \qquad \theta\neq 0$$

In particular,

$$\hat{a}(\xi,\theta) = \frac{i\hat{V}(\theta)}{2\langle \xi,\theta\rangle}, \qquad \langle \xi,\theta\rangle \neq 0.$$

So, we have some more obvious problems when $\langle \xi,\theta\rangle = 0$, e.g.

$$\partial_{x_1}a(x,\xi) = \cos(x_2) \qquad \Leftrightarrow \qquad a(x,\xi) = x_1\cos(x_2) + f(x_2).$$

We need to avoid this situations– resonant zones

5.3. **Gauge transform in dimension 1.** Suppose that $V\in C^\infty(\mathbb{R};\mathbb{R})$ is periodic; i.e.

$$V = \sum_n v_n e^{in\theta}, \qquad v_n = \overline{v_{-n}}, \qquad \sum_n |n|^{2k}|v_n|^2 < \infty$$

Consider

$$H_2 := -\hbar^2\partial_x^2 + \hbar^2\chi(|\hbar D|-1)V\chi(|\hbar D|-1)$$

Lemma 5.3. *There is $\Phi\in\hbar\Psi^{\mathrm{comp}}$ self-adjoint such that*

$$e^{i\Phi}H_2e^{-i\Phi} = -\hbar^2\Delta + \hbar^2 m(\hbar D) + O(\hbar^\infty)_{\Psi^{-\infty}}.$$

Proof. Let $\bar{V} = v_0$ and

$$\phi_0(x,\xi) = \frac{1}{2\xi}\chi(|\xi|-1)^2\int_0^x (V-\bar{V})(s)ds.$$

Then,

$$2\langle \xi,\partial_x\rangle\phi_0 = \chi(|\xi|-1)^2V(x),$$

and, since $V-\bar{V}$ has zero average, $\phi_0\in S^{\mathrm{comp}}$ is periodic in x. Hence,

$$i[Op_h(\hbar\phi_0),\hbar^2\Delta] = \hbar^2\chi(|\hbar D|-1)V\chi(|\hbar D|-1) + \hbar^3\tilde{R}_1,$$

where $\tilde{R}_1 \in \Psi^{\text{comp}}$ periodic in x and $\text{WF}_\text{h}(\tilde{R}_1) \subset \{\xi \,:\, |\xi| - 1 \in \operatorname{supp}\chi\}$ In particular, with $\Phi_0 = Op_h(\hbar\phi_0)$,

$$e^{i\Phi_0}H_2e^{-i\Phi_0} = -\hbar^2\Delta + \hbar^2 m_1(\hbar D) + \hbar^3 R_1,$$

where $m_1 \in S^{\text{comp}}$, $R_1 \in \Psi^{\text{comp}}$ is periodic in x with zero average and $\text{WF}_\text{h}(R_1) \subset \{\xi \,:\, |\xi| - 1 \in \operatorname{supp}\chi\}$. Suppose that we have found ϕ_j such that with $\Phi_{N-1} = \hbar\sum_{j=0}^{N-1}\hbar^{j+1}Op_h(\phi_j)$, we have

$$e^{i\Phi_{N-1}}H_2e^{-i\Phi_{N-1}} = -\hbar^2\Delta + \hbar^2 m_N(\hbar D) + \hbar^{2+N}R_N,$$

where $m_N \in S^{\text{comp}}$, $R_N \in \Psi^{\text{comp}}$ is periodic in x with zero average and $\text{WF}_\text{h}(R_N) \subset \{\xi \,:\, |\xi| - 1 \in \operatorname{supp}\chi\}$.

Now, put

$$\phi_N(x,\xi) = \frac{1}{2\xi}\int_0^x \sigma(R_N)(s,\xi)ds.$$

Then,

$$2\langle\xi,\partial_x\rangle\phi_N = \sigma(R_N),$$

and, since $\sigma(R_N)$ has zero average, $\phi_N \in S^{\text{comp}}$ is periodic in x. Hence,

$$i[Op_h(\hbar^{N+1}\phi_N), \hbar^2\Delta] = \hbar^{N+2}R_N + \hbar^{N+3}\tilde{R}_{N+1},$$

for some $\tilde{R}_{N+1} \in \Psi^{\text{comp}}$ periodic in x with $\text{WF}_\text{h}(\tilde{R}_{N+1}) \subset \{\xi \,:\, |\xi| - 1 \in \operatorname{supp}\chi\}$. In particular,

$$e^{i\Phi_N}H_2e^{-i\Phi_N} = -\hbar^2\Delta + \hbar^2 m_{N+1}(\hbar D) + \hbar^{3+N}R_{N+1},$$

for some $m_{N+1} \in S^{\text{comp}}$, $R_{N+1} \in \Psi^{\text{comp}}$ is periodic in x with zero average and $\text{WF}_\text{h}(R_{N+1}) \subset \{\xi \,:\, |\xi| - 1 \in \operatorname{supp}\chi\}$.

Setting $\Phi \sim \sum_j \hbar^{1+j}Op_h(\phi_j)$, then completes the proof. □

Theorem 5.4. *Let $V \in C^\infty(\mathbb{R};\mathbb{R})$ be 2π periodic. Then, with*

$$H := -\hbar^2\Delta + \hbar^2 V,$$

for ω near 1, we have

$$E(H;(-\infty,\omega])(x,y) = \frac{1}{2\pi\hbar}\int_{\xi_h^-(\omega)}^{\xi_h^+(\omega)} e^{\frac{i}{\hbar}\langle x-y,\xi\rangle}a(x,y,\xi)d\xi,$$

where

$$\xi_h^\pm(\omega) = \sum_j \hbar^j\xi_j^\pm(\omega), \qquad \xi_0^\pm(\omega) = \pm\sqrt{\omega},$$

and $a \in S^0$.

Proof. First, observe that by Corollary 4.7, it is enough to find an asymptotic expansion for the spectral projector of

$$H_2 := -\hbar^2\Delta + \hbar^2\chi(|\hbar D| - 1)V\chi(|\hbar D| - 1).$$

Then, by Lemma 5.3, there are $\Phi \in \hbar\Psi^{\text{comp}}$, $m \in S^{\text{comp}}$ such that

$$e^{i\Phi}H_2e^{-i\Phi} = -\hbar^2\Delta + \hbar^2 m(\hbar D) + O(\hbar^\infty)_{\Psi^{-\infty}}.$$

Let

$$H_3 := e^{-i\Phi}\big(-\hbar^2\Delta + \hbar^2 m(\hbar D)\big)e^{i\Phi}, \qquad \tilde{H}_3 := -\hbar^2\Delta + \hbar^2 m(\hbar D).$$

Then,

$$H_3 - H_2 = O(\hbar^\infty)_{\Psi^{-\infty}}$$

and hence, by Corollary 4.3, it is enough to find a complete asymptotic expansion for the spectral projector of H_3. Since, $\tilde{H}_3$ is a Fourier multiplier, we have by Lemma

$$E(\tilde{H}_3;(-\infty,\omega])(x,y) = \frac{1}{2\pi\hbar}\int_{|\xi|^2+\hbar^2 m(\xi)\leq\omega} e^{\frac{i}{\hbar}\langle x-y,\xi\rangle}d\xi.$$

Now, let $\xi_h^\pm(\omega)$ solve

$$|\xi_h^\pm(\omega)|^2 + \hbar^2 m(\xi_h^\pm(\omega)) = \omega,$$

with $|\pm\sqrt{\omega}-\xi_h^\pm| < \frac{\omega}{2}$. Then, by the implicit function theorem,

$$\xi_h^\pm = \sum_j h^j \xi_j^\pm(\omega), \qquad \xi_0^\pm = \pm\sqrt{\omega},$$

and

$$E(\tilde{H}_3;(-\infty,\omega])(x,y) = \frac{1}{2\pi\hbar}\int_{\xi_h^-(\omega)}^{\xi_h^+(\omega)} e^{\frac{i}{\hbar}\langle x-y,\xi\rangle}d\xi.$$

Next, observe that

$$E(H_3;(-\infty,\omega]) = e^{-i\Phi}E(\tilde{H}_3;(-\infty,\omega])e^{i\Phi},$$

and, since $e^{\pm i\Phi}\in\Psi^0$,

$$e^{\pm i\Phi}(x,y) = \frac{1}{2\pi\hbar}\int e^{\frac{i}{\hbar}\langle x-y,\xi\rangle}a_\pm(x,\xi)d\xi,$$

for some $a_\pm\in S^0$. In particular,

$$E(H_3;(-\infty,\omega])(x,y) = \frac{1}{(2\pi\hbar)^3}\int_{\xi_h^-(\omega)}^{\xi_h^+(\omega)}\int e^{\frac{i}{\hbar}(\langle x-t,\xi\rangle+\langle t-z,\tau\rangle+\langle z-y,\zeta\rangle}a_-(x,\xi)a_+(z,\zeta)dzdtd\xi d\zeta d\tau.$$

Performing stationary phase in (t,z,ξ,ζ) then completes the proof. □

5.4. **Gauge transform in dimension ≥ 2.** We will focus on the gauge transform as it applies to a two dimensional periodic operator. That is, let

$$H = -\hbar^2\Delta + \hbar^2 V,$$

where V is periodic with respect to the lattice Γ, with dual lattice Γ' i.e.

$$V = \sum_{\theta\in\Gamma'} a_\theta e^{i\langle\theta,x\rangle}, \qquad a_\theta = \bar{a}_{-\theta}.$$

We first replace V by $\chi(|\hbar D|-1)V\chi(|\hbar D|-1)$ using Corollary 4.7 as in the 1 dimensional case and claim that it is sufficient to find an asymptotic expansion for

$$H_1 := -\hbar^2\Delta + \hbar^2\chi(|\hbar D|-1)V\chi(|\hbar D|-1),$$

As before, our goal is to find a unitary operator, U so that

$$UH_1U^* = H_2 + O(\hbar^\infty)_{H_\hbar^{-s}\to H_\hbar^s}, \qquad H_2 := \hbar^2\Delta + \hbar^2 m(\hbar D) + \hbar^2 X_{res}$$

where X_{res} is sufficiently structured so that one can compute the spectral function for H_2. As we have seen, in higher dimensions, one cannot avoid resonant zones and so the procedure will be substantially more complicated than in 1-dimension.

One extra complication is that, in practice, one uses a discrete parameter

$$10^{-3}\hbar \leq \hbar_n \leq 10^3\hbar,$$

which is locally constant in $\hbar$. We will then compare the results of our computation with two different choices of $\hbar_n$ in order to obtain the final asymptotics. However, this extra complication is technical rather than conceptual, so we will work with only the small parameter $\hbar$.

Lemma 5.5. *Suppose that $u, v \in \Gamma \setminus \{0\}$ with $u \notin \mathbb{R}v$. Then, $|\sin\theta| \geq c\langle \|u\|\|v\|\rangle^{-1}$, where θ is the angle between u and v.*

Proof. The volume of the parallelogram spanned by u, v is given by

$$\det\begin{pmatrix} u & v \end{pmatrix} = \det FA,$$

where A has integer coefficients and F is invertible.

$$|\det\begin{pmatrix} u & v \end{pmatrix}| \geq c.$$

Now,

$$\det\begin{pmatrix} u & v \end{pmatrix} = \|u\|\|v\|\sin\theta,$$

where θ is the angle between u and v. Therefore

$$|\sin\theta| \geq \frac{c}{\|u\|\|v\|} \geq \frac{c}{\langle \|u\|\|v\|\rangle}.$$

□

Lemma 5.6. *Let $0 < \delta < \frac{1}{3}$, $2\delta < \varepsilon < \min(\frac{1}{2}, 1-\delta)$. Then there is $\Phi \in h^{1-\varepsilon}\Psi_\varepsilon^{\mathrm{comp}}$ such that*

$$e^{i\Phi}H_1e^{-i\Phi} = -\hbar^2\Delta + \hbar^2 m(\hbar D) + \sum_{|\theta|\leq \hbar^{-\delta}} \hbar^2 Op_h(e_\theta(\xi)e^{i\theta x}) + O(\hbar^\infty)_{\Psi^{-\infty}},$$

where $m, e_\theta \in S_\varepsilon^{\mathrm{comp}}$,

$$\operatorname{supp} e_\theta \subset \{(x,\xi) \,:\, |h^{-\varepsilon}\langle \xi, \theta\rangle| \leq 3,\, |\xi| - 1 \in \operatorname{supp}\chi\},$$

and,

$$e_\theta, m \text{ are analytic in } \langle \xi, \theta^\perp\rangle \text{ for } |\xi| \sim 1 \text{ and } |h^{-\varepsilon}\langle \xi, \theta\rangle| \leq 3.$$

Proof. Fix $\delta > 0$ and let

$$V_h = \sum_{\substack{\theta\in\Gamma \\ |\theta|<\hbar^{-\delta}}} v_\theta e^{i\langle \theta, x\rangle}.$$

Then,

$$\|V - V_h\|_{C^N} \leq C_N \hbar^N.$$

Let $\psi \in C_c^\infty(-3,3)$ with $\psi \equiv 1$ on $[-2,2]$. Then, let $\psi_j \in C_c^\infty(-2,2)$, $j = 1, 2, \dots$ with $\psi_1 \equiv 1$ on $[-1,1]$ and $\operatorname{supp}\psi_j \cap \operatorname{supp}(1-\psi_{j+1}) = \emptyset$. Now, let $\varepsilon > 0$, with $\varepsilon + \delta < 1$, and set $\bar{V} = v_0$ and put

$$\phi_0 = \sum_{|\theta|\leq \hbar^{-\delta}} \phi_{0,\theta}(x,\xi),$$

with

$$\phi_{0,\theta}(x,\xi)=\chi^2(|\xi|-1)(1-\psi_1(h^{-\delta}\langle\xi,\theta\rangle))\int_0^\infty v_\theta e^{i\langle\theta,x-2t\xi\rangle}dt$$
$$=\chi^2(|\xi|-1)(1-\psi_1(h^{-\delta}\langle\xi,\theta\rangle))v_\theta\frac{e^{i\langle\theta,x\rangle}}{2i\langle\xi,\theta\rangle},$$

so that

$$2\langle\xi,\partial_x\phi_{0,\theta}\rangle=\chi^2(|\xi|-1)(1-\psi_1(h^{-\delta}\langle\xi,\theta\rangle))v_\theta e^{i\theta x}.$$

In particular, $\phi_0\in\hbar^{-\varepsilon}S_\varepsilon^{\mathrm{comp}}$ and, with $\Phi_0=\hbar Op_h(\phi_0)$,

$$e^{i\Phi_0}He^{-i\Phi_0}=-\hbar^2\Delta+\hbar^2\chi^2(|\hbar D|-1)v_0+\sum_{|\theta|\leq\hbar^{-\delta}}\hbar^2\chi(|\hbar D|-1)\psi_1(\hbar^{-\varepsilon}\langle\theta,\hbar D\rangle)v_\theta e^{i\langle\theta,x\rangle}\chi(|\hbar D|-1)$$
$$+\hbar^{3-\varepsilon}R_1+O(\hbar^\infty)_{\Psi^{-\infty}},$$

where $R_1\in\Psi_\varepsilon$ is Γ periodic and $\mathrm{WF_h}(R_1)\subset\{\xi\,:|\xi|-1\in\operatorname{supp}\chi\}$.

Now, suppose $\phi_j\in S_\varepsilon^{\mathrm{comp}}$ such that, with $\Phi_{N-1}=\sum_{j=0}^{N-1}\hbar\hbar^{j(1-\varepsilon)}\phi_j$, such that

$$e^{i\Phi_{N-1}}He^{-i\Phi_{N-1}}=-\hbar^2\Delta+\hbar^2m_N(\hbar D)+\sum_{|\theta|\leq\hbar^{-\delta}}\hbar^2Op_h(e_{\theta,N}(\xi)e^{i\theta x})+\hbar^{2+N(1-\varepsilon)}R_N+O(\hbar^\infty)_{\Psi^{-\infty}},$$

where

$$\operatorname{supp}e_{\theta,N}\subset\{\xi\,:\,h^{-\varepsilon}\langle\xi,\theta\rangle\in\operatorname{supp}\psi_N,\,|\xi|-1\in\operatorname{supp}\chi\},$$

and, with $R_N=Op_h(r_N)$, $r_N=\sum_\theta r_{\theta,N}(\xi)e^{i\langle\theta x\rangle}$, we have

$$e_{\theta,N},m_N,,\,r_{\theta,N}\text{ are analytic }\langle\xi,\theta^\perp\rangle\text{ for }|\xi|\sim 1,|h^{-\varepsilon}\langle\xi,\theta\rangle|\leq 3$$

and

$$\operatorname{supp}r_{\theta,N}\subset\{\xi\,:|\xi|-1\in\operatorname{supp}\chi\}$$

Now, put

$$\phi_N=\sum_{0<|\theta|\leq\hbar^{-\delta}}\phi_{N,\theta}(x,\xi),$$

with

$$\phi_{N,\theta}(x,\xi)=(1-\psi_{N+1}(h^{-\delta}\langle\xi,\theta\rangle))\int_0^\infty r_{N,\theta}(\xi)e^{i\langle\theta,x-2t\xi\rangle}dt$$
$$=(1-\psi_N(h^{-\delta}\langle\xi,\theta\rangle))r_{N,\theta}(\xi)\frac{e^{i\langle\theta,x\rangle}}{2i\langle\xi,\theta\rangle},$$

we obtain

$$e^{i\Phi_N}He^{-i\Phi_N}=-\hbar^2\Delta+\hbar^2m_{N+1}(\hbar D)+\sum_{|\theta|\leq\hbar^{-\delta}}\hbar^2Op_h(e_{\theta,N+1}e^{i\theta x})+\hbar^{2+(N+1)(1-\varepsilon)}R_{N+1}+O(\hbar^\infty)_{\Psi^{-\infty}},$$

where $m_{N+1}\in S_\varepsilon^{\mathrm{comp}}$, $e_{\theta,N+1}\in S_\varepsilon^{\mathrm{comp}}$, $R_{N+1}\in S_\varepsilon^{\mathrm{comp}}$,

$$\operatorname{supp}e_{\theta,N_1}\subset\{(x,\xi)\,:\,h^{-\varepsilon}\langle\xi,\theta\rangle\in\operatorname{supp}\psi_{N+1}\},$$

and, with $R_{N+1} = Op_h(r_{N+1})$, $r_{N+1}(x,\xi) = \sum_\theta r_{\theta,N+1}(\xi)e^{i\langle\theta x\rangle}$, we have

$$e_{\theta,N+1},\ r_{\theta,N+1}, m_{N+1} \text{ are analytic } \langle\xi,\theta^\perp\rangle \text{ for } |\xi|\sim 1, |h^{-\varepsilon}\langle\xi,\theta\rangle| \leq 3$$

and

$$\operatorname{supp} r_{\theta,N+1} \subset \{\xi : |\xi|-1 \in \operatorname{supp}\chi\}$$

Putting $\Phi \sim \hbar\sum_j \hbar^{j(1-\varepsilon)}\phi_j$ completes the proof. □

6. Computation of the integrated density of states

For simplicity, we now restrict our attention to the integrated density of states:

$$N(H,\lambda) := \frac{1}{(2\pi)^d}\int_{k\in\Gamma^*} N(H,\lambda,k)dk,$$

where $N(H,\lambda,k)$ is the eigenvalue counting function for

$$e^{ikx}He^{-ikx}, \qquad u(x+\gamma) = e^{ik\gamma}u(x).$$

In particular, this quantity is clearly unitarily invariant (provided the unitary operators also commute with translation by Γ). Therefore, it is enough to compute $N(H_2,\lambda)$, where

$$H_2 := -h^2\Delta + h^2 m(hD) + \sum_{|\theta|\leq h^{-\delta}} h^2 Op_h(e_\theta(\xi)e^{i\theta x}),$$

and we will focus on this from now on. To simplify things further, we will assume there is exactly one resonant zone corresponding to $\theta = (1,0)$. In particular,

$$H_2 := -h^2\Delta + h^2 m(hD) + \sum_{n=1}^{h^{-\delta}} h^2 Op_h(e_n(\xi)e^{inx_1}),$$

where $e_n \in S_\varepsilon^{\mathrm{comp}}$ is supported near $|\xi| = 1$ and in $|\xi_1| < h^\varepsilon$ and is analytic in ξ_2 in a neighborhood of $|\xi| = 1$. We then define

$$\mathscr{D} := \{|\xi_1| \geq Mh^\varepsilon\}, \qquad \mathscr{R} := \{|\xi_1| \leq Mh^\varepsilon\},$$

and put

$$\Pi_{\mathscr{D}} := \mathcal{F}_h^{-1}1_{\mathscr{D}}\mathcal{F}_h$$

and

$$\Pi_{\mathscr{R}} := \mathcal{F}_h^{-1}1_{\mathscr{R}}\mathcal{F}_h,$$

It will also be conventient to use the fact that

$$N(H,\omega) = \lim_{R\to\infty}\frac{1}{\pi R^2}\int_{|x_0|\leq R} E(H,\omega,x_0,x_0)dx_0$$

Observe that

$$H_2\Pi_{\mathscr{D}} = \Pi_{\mathscr{D}}H_2,$$

and

$$H_2\Pi_{\mathscr{R}} = \Pi_{\mathscr{R}}H_2.$$

Therefore,

$$e(H_2,\omega,x_0,x_0) = \langle E(H_2,\omega)\delta_{x_0},\delta_{x_0}\rangle = \langle E(H_2,\omega)\Pi_{\mathscr{R}}\delta_{x_0},\Pi_{\mathscr{R}}\delta_{x_0}\rangle + \langle E(H_2,\omega)\Pi_{\mathscr{D}}\delta_{x_0},\Pi_{\mathscr{D}}\delta_{x_0}\rangle.$$

Define

$$N_{\mathscr{D}}(H_2,\omega) = \lim_{R\to\infty}\frac{1}{\pi R^2}\int_{|x_0|\leq R}\langle E(H_2,\omega)\Pi_{\mathscr{D}}\delta_{x_0},\Pi_{\mathscr{D}}\delta_{x_0}\rangle dx_0,$$
$$N_{\mathscr{R}}(H_2,\omega) := \lim_{R\to\infty}\frac{1}{\pi R^2}\int_{|x_0|\leq R}\langle E(H_2,\omega)\Pi_{\mathscr{R}}\delta_{x_0},\Pi_{\mathscr{R}}\delta_{x_0}\rangle dx_0.$$

We first understand the contribution of the non-resonant zones to the integrated density of states

Lemma 6.1. *We have*

$$N_{\mathscr{D}}(H_2,\omega) = \int_{\substack{|\eta|^2+h^2m(\eta)\leq\omega\\ \eta\in\mathscr{D}}} d\eta.$$

Proof. Observe that

$$E(H_2,\omega)\Pi_{\mathscr{D}} = E(\Pi_{\mathscr{D}}H_2\Pi_{\mathscr{D}},\omega)$$

and hence the kernel of $E(H_2,\omega)\Pi_{\mathscr{D}}$ is given by

$$\frac{1}{(2\pi h)^2}\int_{1_{\mathscr{D}}(\eta)(|\eta|^2+h^2m(\eta))\leq\omega} e^{\frac{i}{h}\langle x-y,\eta\rangle}.$$

Now, we compute

$$\Pi_{\mathscr{D}}\delta_{x_0} = \frac{1}{(2\pi h)^2}\int_{\mathscr{D}} e^{\frac{i}{h}\langle x-x_0,\xi\rangle}d\xi.$$

So that

$$\begin{aligned}
&\langle E(H_2,\omega)\Pi_{\mathscr{D}}\delta_{x_0},\Pi_{\mathscr{D}}\delta_{x_0}\rangle\\
&=\frac{1}{(2\pi h)^6}\int_{1_{\mathscr{D}}(\eta)(|\eta|^2+h^2m(\eta))\leq\omega}\int_{\xi\in\mathscr{D}}\int_{\zeta\in\mathscr{D}}\int e^{\frac{i}{h}(\langle x-y,\eta\rangle+\langle y-x_0,\xi\rangle-\langle x-x_0,\zeta\rangle)}dxdyd\zeta d\xi d\eta\\
&=\frac{1}{(2\pi h)^4}\int_{\substack{1_{\mathscr{D}}(\eta)(|\eta|^2+h^2m(\eta))\leq\omega\\ \eta\in\mathscr{D}}}\int_{\zeta\in\mathscr{D}}\int e^{\frac{i}{h}(\langle x-x_0,\eta\rangle-\langle x-x_0,\zeta\rangle)}dxd\zeta d\eta\\
&=\frac{1}{(2\pi h)^2}\int_{\substack{|\eta|^2+h^2m(\eta)\leq\omega\\ \eta\in\mathscr{D}}}d\eta,
\end{aligned}$$

and the claim follows since $\int_{|x_0|\leq R}1dx_0 = \pi R^2$. □

Now, to understand $N_{\mathscr{R}}(H_2,\omega)$, we conjugate by the Fourier transform. In particular,

$$\tilde{H}_2 := \mathcal{F}H_2\mathcal{F}^{-1} = |\xi|^2 + h^2m(\xi) + \mathcal{F}\sum_n Op_h(e_n(\eta)e^{inx_1})\mathcal{F}^{-1}.$$

We next show that the last operator on the right hand side nearly acts as a shift.

Lemma 6.2. *The kernel, $K(\xi,\eta)$ of $\mathcal{F}Op_h(e_ne^{inx_1})\mathcal{F}$ is given by*

$$e_n(\xi - nh\tfrac{e_1}{2})\delta(\eta-\xi+nhe_1).$$

Proof. We have

$$\begin{aligned}
K(\xi,\eta) &= \frac{1}{(2\pi h)^4}\int e^{\frac{i}{h}(-\langle \xi,x\rangle+\langle x-y,\zeta\rangle+\langle \eta,y\rangle)}e_n(\zeta)e^{\frac{ni}{2}(x_1+y_1)}dxdyd\zeta\\
&= \frac{1}{(2\pi h)^4}\int e^{\frac{i}{h}(\langle \zeta-\xi+h\frac{n}{2}e_1,x\rangle+\langle y,\eta-\zeta+h\frac{n}{2}e_1\rangle}e_n(\zeta)dxdyd\zeta\\
&= \frac{1}{(2\pi h)^2}\int e^{\frac{i}{h}(\langle y,\eta-\xi+hne_1\rangle)}e_n(\xi-h\tfrac{n}{2}e_1)dy\\
&= e_n(\xi-h\tfrac{n}{2}e_1)\delta(\eta-\xi+hne_1).
\end{aligned}$$

□

Now, we write $1_{\mathscr{R}}\tilde{H}_2 1_{\mathscr{R}}$ as a direct integral

$$\int_0^h \tilde{H}_2(k)dk$$

To realize this, we let $I := \{|\xi_1| \leq Mh^{\varepsilon}\}$ and $U : L^2(\mathbb{R}\times I) \to L^2(\mathbb{R}; L^2([0,h])^N)$, where $N = \lceil Mh^{\varepsilon-1}\rceil$ with

$$[U(k,\xi_2)f]_j = f(k+jh,\xi_2)$$

and adjoint

$$[U^*g](\xi_1,\xi_2) = g(\xi_1 - j(\xi_1), j(\xi_1), \xi_2),$$

where

$$j(\xi_1) := \{j \,:\, \xi_1 \in jh + [0,h)\}.$$

One can then check that U is unitary and

$$U(k,\xi_2)1_{\mathscr{R}}\tilde{H}_2 1_{\mathscr{R}}u = \tilde{H}_{\mathscr{R}}(k,\xi_2)U(k,\xi_2)u,$$

where $\tilde{H}_{\mathscr{R}}(k,\xi_2)$ is the $2N\times 2N$ matrix

$$\xi_2^2 I + \mathrm{diag}(k+jh)^2 + h^2 e_{i-j}(k+\tfrac{i-j}{2}h,\xi_2)$$

Lemma 6.3. *We have*

$$N_{\mathscr{R}}(H_2,\omega) = \frac{1}{(2\pi h)^2}\int\int_0^h \mathrm{tr}(E(\tilde{H}_{\mathscr{R}}(k,\xi_2),\omega))dkd\xi_2$$

Proof. Now, let $x_0 \in \mathbb{R}^2$, put $x_s := x_0 + se_1$

$$\begin{aligned}
&\frac{1}{2\pi h}\int_0^{2\pi h}\langle E(H_2,\omega)\Pi_{\mathscr{R}}\delta_{x_s},\Pi_{\mathscr{R}}\delta_{x_s}\rangle\\
&=\frac{1}{(2\pi h)^3}\int_0^{2\pi h}\langle E(1_{\mathscr{R}}\tilde{H}_2 1_{\mathscr{R}},\omega)1_{\mathscr{R}}(\xi)e^{-\frac{i}{h}\langle \xi,x_s\rangle},1_{\mathscr{R}}(\xi)e^{-\frac{i}{h}\langle \xi,x_s\rangle}\rangle\\
&=\frac{1}{(2\pi h)^3}\int_0^{2\pi h}\langle E(\tilde{H}_{\mathscr{R}}(k,\xi_2),\omega)U1_{\mathscr{R}}(\xi)e^{-\frac{i}{h}\langle \xi,x_s\rangle},U1_{\mathscr{R}}(\xi)e^{-\frac{i}{h}\langle \xi,x_s\rangle}\rangle_{\xi_2,k}ds\\
&=\frac{1}{(2\pi h)^3}\int_0^{2\pi h}\int\int_0^h\sum_{\ell j}(E(\tilde{H}_{\mathscr{R}}(k,\xi_2),\omega))_{\ell j}1_{\mathscr{R}}(k+jh,\xi_2)e^{-\frac{i}{h}((k+jh)((x_0)_1+s)}\\
&\qquad 1_{\mathscr{R}}(k+\ell h,\xi_2)e^{\frac{i}{h}((k+\ell h)((x_0)_1+s)}dkd\xi_2ds\\
&=\frac{1}{(2\pi h)^2}\int\int_0^h\sum_j(E(\tilde{H}_{\mathscr{R}}(k,\xi_2),\omega))_{jj}1_{\mathscr{R}}(k+jh,\xi_2)dkd\xi_2\\
&=\frac{1}{(2\pi h)^2}\int\int_0^h\operatorname{tr}(E(\tilde{H}_{\mathscr{R}}(k,\xi_2),\omega))dkd\xi_2.
\end{aligned}$$

□

Now, we crucially use monotonicity of $\tilde{H}_{\mathscr{R}}(k,\xi_2)$ as a function of ξ_2 for ξ_2 near 1. Indeed, observe that

$$\operatorname{tr}(E(\tilde{H}_{\mathscr{R}}(k,\xi_2),\omega))=\#\{\lambda\in\operatorname{Spec}(\tilde{H}_{\mathscr{R}}(k,\xi_2))\,:\,\lambda\le\omega\}.$$

But, since $\tilde{H}_{\mathscr{R}}(k,\xi_2)$ is monotone, all of its eigenvalues are and hence, letting $\pm\tau_j^{\pm}(k)$ be the unique solution near $\xi_2=\pm\sqrt{\omega}$ such that

$$\lambda_j(\tilde{H}_{\mathscr{R}}(k,\pm\tau_j^{\pm}(k)))=\omega,$$

we have

$$\frac{1}{(2\pi h)^2}\int\int_0^h\operatorname{tr}(E(\tilde{H}_{\mathscr{R}}(k,\xi_2),\omega))dkd\xi_2=\sum_{j,\pm}\frac{1}{(2\pi h)^2}\int_0^h\tau_j^{\pm}(k)dk.$$

Now, let $\gamma_\pm := \{|z\mp\sqrt{\omega}|<r_0\}$ with $r_0 \ll 1$ and counter-clockwise orientation observe that the number of zeros of $f(z,k,\omega):=(\det\tilde{H}_{\mathscr{R}}(k,z)-\omega)$ inside γ is given by

$$\begin{aligned}
\#(f,\gamma_\pm)&=\frac{1}{2\pi i}\int_{\gamma_\pm}\frac{f'(z)}{f(z)}dz\\
&=\frac{1}{2\pi i}\int_{\gamma_\pm}\operatorname{tr}(\tilde{H}_{\mathscr{R}}(k,z))'(\tilde{H}_{\mathscr{R}}(k,z)-\omega)^{-1}dz\\
&=\frac{1}{2\pi i}\int_{\gamma_\pm}\operatorname{tr}(2zI+O(h^{2-\varepsilon}))(z^2I+O(h^{2-2\varepsilon})-\omega)^{-1}dz\\
&=\frac{1}{2\pi i}\int_{\gamma_\pm}\operatorname{tr}2zI(z^2-\omega)^{-1}dz+O(h^{2-2\varepsilon})=2N,
\end{aligned}$$

since the integral is integer valued. In particular, the only zeros of f inside $\gamma_\pm$ are the $2N$ zeros on the real axis. Let $E_{\mathscr{R}}(k,z) := \tilde{H}_{\mathscr{R}}(k,z) - z^2 I$. Then we have

$$\begin{aligned}
\sum_j \tau_j^\pm(k) &= \frac{1}{2\pi i}\int_{\gamma_\pm} \pm z(\det \tilde{H}_{\mathscr{R}}(k,z)-\omega)'(\det \tilde{H}_{\mathscr{R}}(k,z)-\omega)^{-1}dz \\
&= \frac{1}{2\pi i}\int_{\gamma_\pm} \pm \operatorname{tr}(2z^2 I + zE'_{\mathscr{R}}(k,z))(z^2I + E_{\mathscr{R}}(k,z)-\omega)^{-1}dz \\
&= \frac{1}{2\pi i}\int_{\gamma_\pm} \pm \operatorname{tr}(2z^2 I + zE'_{\mathscr{R}}(k,z))(z^2-\omega)^{-1}(I+(z^2-\omega)^{-1}E_{\mathscr{R}}(k,z))^{-1}dz \\
&= \frac{1}{2\pi i}\int_{\gamma_\pm} \pm \operatorname{tr}(2z^2 I + zE'_{\mathscr{R}}(k,z))\sum_\ell (-1)^\ell (z-\sqrt{\omega})^{-\ell-1}(z+\sqrt{\omega})^{-\ell-1}E_{\mathscr{R}}(k,z)^\ell dz \\
&= \pm\sum_\ell \frac{1}{\ell!}\frac{d^\ell}{d\xi_2^\ell}\operatorname{tr}(2\xi_2^2 I + \xi_2 E'_{\mathscr{R}}(k,\xi_2))(-1)^\ell(\xi_2\pm\sqrt{\omega})^{-\ell-1}E_{\mathscr{R}}(k,\xi_2)^\ell\Big|_{\xi_2=\pm\sqrt{\omega}}.
\end{aligned}$$

Hence, since the right-hand side has a complete asymptotic expansion in h with coefficients that are smooth functions of k, the proof is complete.

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