The operator curl: a case study in the spectral theory of systems. II

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Proper definition of curl and its basic properties

It is natural to define curl as an operator

$$\text{curl} : \delta\Omega^2 \cap H^1 \to \delta\Omega^2,$$

where $\delta\Omega^2$ is the Hilbert space of real-valued coexact 1-forms with inner product

$$\langle u, v \rangle := \int_M *u \wedge v = \int_M u \wedge *v$$

and H^1 is the Sobolev space of real-valued 1-forms which are square integrable together with their first partial derivatives.

Theorem 1

(a) The operator curl is self-adjoint.

(b) The spectrum of curl is discrete and accumulates to $+\infty$ and to $-\infty$.

(c) Zero is not an eigenvalue of curl.

(d) The operator curl^{-1} is a bounded operator from $\delta\Omega^2 \cap H^s$ to $\delta\Omega^2 \cap H^{s+1}$ for all $s \geq 0$.

Classical approach to spectral asymmetry

Definition of *eta function:*

$$\eta_{\mathrm{curl}}(s) := \sum_{\lambda_k \neq 0} \frac{\operatorname{sgn} \lambda_k}{|\lambda_k|^s}\,.$$

Series converges absolutely for $\operatorname{Re} s > 3$.

Meromorphic continuation to $\mathbb{C}$.

Eta function generalises the more familiar zeta function.

Definition of *eta invariant:* $\eta_{\mathrm{curl}}(0)$.

Mathematicians who contributed to this subject area: Atiyah, Patodi, Singer, Hitchin, Gilkey, Pontryagin, Hirzebruch, Chern, Simons, Seeley ...

Key words: Hirzebruch L-polynomials, Hirzebruch $\hat{A}$-polynomials, Pontryagin forms, Pontryagin classes ...

Our approach to spectral asymmetry

Put

$$P_\pm := \theta(\pm\,\mathrm{curl})\,,$$

where

$$\theta(x) := \begin{cases} 1, & x > 0, \\ 0, & x \le 0, \end{cases}$$

is the Heaviside step function.

Then, morally,

$$\eta_{\mathrm{curl}}(0) = \mathrm{Tr}(P_+ - P_-)\,.$$

Major problem: operator $P_+ - P_-$ is not of trace class.

Invariant calculus for operators acting on 1-forms

Definition of subprincipal symbol for operators acting on 1-forms.

Original definition of subprincipal symbol for scalar operators acting on half-densities is due to Duistermaat and Hörmander (1972).

Subprincipal symbol of adjoint.

Subprincipal symbol of composition.

In dimension d the subprincipal symbol of Hodge Laplacian is zero.

In dimension 3 the subprincipal symbol of curl is zero.

Constructing the projection operators $P_\pm$

Theorem 2 The operators P_+ and P_- are pseudodifferential operators of order zero and we have written down **explicitly** the homogeneous components of their symbols of degree of homogeneity $0, -1, -2, -3$.

Components of degree of homogeneity -2 and -3 have been written in geodesic normal coordinates.

Algorithm is described in our paper

Capoferri and Vassiliev, *Invariant subspaces of elliptic systems I: pseudodifferential projections*, Journal of Functional Analysis, 2022.

Algorithm is global and does not use local coordinates. Magic!

Implementation of 'magic' algorithm benefits from the use of the computer algebra package Mathematica©.

Where did the 'magic' algorithm come from?

Spectral theory of elliptic systems. Second Weyl coefficient.

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2020: Matteo Capoferri and I realised that we have been looking at elliptic systems the wrong way. Should look for almost invariant subspaces and pseudodifferential projections. Benefit of hindsight.

Elephant in the room

How do we calculate the trace of the operator $P_+ - P_-$?

The operator $P_+ - P_-$ is of order 0. Trace class requires order to be strictly less than -3 .

Calculating the trace of an operator acting on 1-forms

Consider

$$Q: \ u_\alpha(x) \mapsto \int_M \mathfrak{q}_\alpha{}^\beta(x,y)\, u_\beta(y)\, \rho(y)\, \mathrm{d}y\,,$$

where $\mathfrak{q}$ is the (distributional) integral kernel (Schwartz kernel) and ρ is the Riemannian density.

Suppose that the integral kernel $\mathfrak{q}$ is sufficiently smooth. Then

$$\operatorname{Tr} Q = \int_M \mathfrak{q}_\alpha{}^\alpha(x,x)\, \rho(x)\, \mathrm{d}x\,,$$

Idea: split the process of calculating trace into two separate steps.

- Take matrix trace first, which would give a scalar operator.
- Calculate the trace of the scalar operator the usual way, by taking the value of the integral kernel on the diagonal $x = y$ and integrating over the manifold M.

Matrix trace of an operator acting on 1-forms

Definition 3 The matrix trace of an operator acting on 1-forms is the scalar operator obtained by contracting tensor indices in the integral kernel $\mathfrak{q}_\alpha{}^\beta(x,y)$ of the original operator. No assumptions on the smoothness of the integral kernel.

Slight problem: tensor indices α and β live at different points, x and y. To make above definition invariant need to perform parallel transport along shortest geodesic connecting x and y.

Another minor problem: need smooth cut-off about the diagonal $x = y$ so that the shortest geodesic connecting x and y is unique.

Matrix trace of an operator acting on 1-forms is defined uniquely modulo the addition of a scalar operator whose integral kernel is infinitely smooth and vanishes in a neighbourhood of the diagonal.

Properties of matrix trace of operator acting on 1-forms

Matrix trace of adjoint.

Principal and subprincipal symbols of matrix trace.

Matrix trace of a differential operator is a differential operator.

In dimension d the matrix trace of the Hodge Laplacian is $d\Delta + \frac{1}{3}\,\mathrm{Sc}$, where Δ is the Laplace–Beltrami operator and Sc is scalar curvature.

In dimension 3 the matrix trace of curl is zero.

In dimension 3 the matrix trace of curl^3 is zero.

The asymmetry operator

Definition 4 The *asymmetry operator* A is defined as the matrix trace of the operator $P_+ - P_-$.

The asymmetry operator is a self-adjoint scalar pseudodifferential operator determined by the Riemannian 3-manifold (M, g) and its orientation.

The miracle

Theorem 5 The asymmetry operator is a pseudodifferential operator of order -3.

Reason for miracle: symmetries of the Riemannian curvature tensor

Corollary 6 The asymmetry operator is *almost* trace class.

Singularity of the integral kernel of the asymmetry operator

Theorem 7 The principal symbol of the asymmetry operator reads

$$A_{\mathrm{prin}}(x,\xi) = -\frac{\varepsilon^{\alpha\beta\gamma}}{2\,\rho(x)\,\|\xi\|^5}\,\nabla_\alpha \operatorname{Ric}_\beta{}^\delta(x)\,\xi_\gamma\,\xi_\delta\,,$$

where Ric is the Ricci curvature tensor, $\nabla \operatorname{Ric}$ is its covariant derivative and ε is the totally antisymmetric symbol (Levi-Civita symbol), $\varepsilon^{123} := +1$.

Corollary 8 The singularity of the integral kernel $\mathfrak{a}(x,y)$ of the asymmetry operator is very weak. Namely, $\mathfrak{a}(x,y)$ is a bounded function, smooth outside the diagonal and discontinuous on the diagonal: for any $x \in M$ the limit $\lim_{y\to x}\mathfrak{a}(x,y)$ depends on the direction along which y tends to x.

The regularised local trace of the asymmetry operator

Denote by $\mathbb{S}_\epsilon(x)$ the geodesic sphere of radius $\epsilon > 0$ centred at the point $x \in M$.

Theorem 9 For any $x \in M$ the limit

$$\lim_{\epsilon \to 0^+} \frac{1}{4\pi\epsilon^2} \int_{\mathbb{S}_\epsilon(x)} \mathfrak{a}(x,y)\, \mathrm{d}S_y$$

exists and defines a scalar continuous function

$$\psi^{\mathrm{loc}}_{\mathrm{curl}}(x)\,, \quad \psi^{\mathrm{loc}}_{\mathrm{curl}} : M \to \mathbb{R}\,.$$

Definition 10 We call $\psi^{\mathrm{loc}}_{\mathrm{curl}}(x)$ *the regularised local trace of the asymmetry operator*.

The regularised global trace of the asymmetry operator

Definition 11 We call the number

$$\psi_{\mathsf{curl}} := \int_M \psi_{\mathsf{curl}}^{\mathrm{loc}}(x)\,\rho(x)\,\mathrm{d}x$$

the regularised global trace of the asymmetry operator.

Reconciling our approach with the classical one

Using microlocal techniques, we have defined a differential geometric invariant ψ_{curl}, a measure of the asymmetry of our Riemannian 3-manifold under change of orientation.

Is it true that $\psi_{\mathrm{curl}} = \eta_{\mathrm{curl}}(0)$?

Yes, it is.

Capoferri and Vassiliev, *A microlocal pathway to spectral asymmetry: curl and the eta invariant*, in preparation.

The parameter-dependent asymmetry operator $A^{(s)}$

Put

$$A^{(s)} := \mathfrak{tr}\,[(P_+ - P_-)(-\mathbf{\Delta})^{-s/2}]\,, \qquad s \in \mathbb{R}.$$

$\mathbf{\Delta}$ is the Hodge Laplacian on 1-forms and $\mathfrak{tr}$ is the matrix trace.

Observation 1

$$A^{(0)} = A\,,$$

where A is our original asymmetry operator.

Observation 2 For $s > 3$ we have

$$\mathrm{Tr}\,[A^{(s)}] = \eta_{\mathrm{curl}}(s)\,.$$

Need to perform analytic continuation from $s \in (3, +\infty)$ to $s \in (0, +\infty)$ and carefully examine what happens when $s \to 0^+$.

Properties of the operator $A^{(s)}$

Theorem 12 The operator $A^{(s)}$ is a self-adjoint scalar pseudodifferential operator of order $-s-3$.

Corollary 13 For $s>0$ the operator $A^{(s)}$ is of trace class.

Theorem 14 The principal symbol $[A^{(s)}]_{\mathrm{prin}}$ of the operator $A^{(s)}$ is

$$[A^{(s)}]_{\mathrm{prin}}(x,\xi)=\frac{(s+1)(s+3)}{3\|\xi\|^s}\,A_{\mathrm{prin}}(x,\xi)\,,$$

where A_{prin} is the principal symbol of our original asymmetry operator A.

Weyl coefficients for the operator curl

$$N^{\pm}_{\mathrm{curl}}(\lambda) := \begin{cases} 0 & \text{for} \quad \lambda \le 0, \\ \displaystyle\sum_{0<\pm\lambda_k<\lambda} 1 & \text{for} \quad \lambda > 0. \end{cases}$$

Let $\widehat{\mu} : \mathbb{R} \to \mathbb{C}$ be a smooth function such that $\widehat{\mu} = 1$ in some neighbourhood of the origin and $\operatorname{supp} \widehat{\mu} \subset (-T_0, T_0)$, where T_0 is the infimum of lengths of all the geodesic loops originating from all the points of the manifold. Let μ be the inverse Fourier transform of $\widehat{\mu}$. Then

$$((N^{\pm}_{\mathrm{curl}})' * \mu)(\lambda) = c^{\pm}_2 \lambda^2 + c^{\pm}_1 \lambda + c^{\pm}_0 + c^{\pm}_{-1} \lambda^{-1} + \dots,$$

as $\lambda \to +\infty$. Here the star stands for convolution in the variable λ.

We call the coefficients $c^{\pm}_k$ *Weyl coefficients*.

Residues of the eta function are expressed via Weyl coefficients:

$$\operatorname{Res}(\eta_{\mathrm{curl}}, n) = c_{n-1}^{+} - c_{n-1}^{-}\,, \qquad n = 3, 2, 1, 0, -1, -2, \ldots.$$

Theorem 15 We have

$$c_2^{\pm} = \frac{1}{2\pi^2}\,\mathrm{Vol}(M)\,,$$

$$c_1^{\pm} = 0\,,$$

$$c_0^{\pm} = -\frac{1}{12\pi^2}\int_M \mathrm{Sc}(x)\,\rho(x)\,\mathrm{d}x\,,$$

where $\mathrm{Vol}(M)$ is the Riemannian volume of the manifold, Sc is scalar curvature and ρ is the Riemannian density.

Bracchi, Capoferri and Vassiliev, *Higher order Weyl coefficients for the operator curl*, in preparation.

Zeros of $\eta_{\mathsf{curl}}(s)$?

We have

$$[A^{(s)}]_{\mathrm{prin}}(x,\xi) = 0 \quad \text{for} \quad s=-1 \quad \text{and} \quad s=-3.$$

Furthermore, we have

$$A^{(s)} = 0 \quad \text{for} \quad s=-1 \quad \text{and} \quad s=-3.$$

Possible zeros of $\eta_{\mathsf{curl}}(s)$:

$$s=-1, \qquad s=-3.$$