B3D Handout 10: Laplacian in Operator Form.

We will use operator notation to prove the form of the Laplacian ∇^2 in plane polar coordinates. Remember the extended chain rule: if $x = x(s,t)$ and $y = y(s,t)$, and we have a function $f(x,y)$ so that

$$F(s,t) = f(x(s,t), y(s,t))$$

then

$$D_s = \frac{\partial x}{\partial s} D_x + \frac{\partial y}{\partial s} D_y \qquad D_t = \frac{\partial x}{\partial t} D_x + \frac{\partial y}{\partial t} D_y.$$

In plane polar coordinates, let us use s for r and t for θ:

$$x = s\cos t \qquad y = s\sin t$$

then the extended chain rule gives us

$$\begin{aligned} D_s &= \cos t D_x + \sin t D_y \\ D_t &= -s\sin t D_x + s\cos t D_y. \end{aligned}$$

We will start from the standard form:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

which gives us the linear operator

$$L \quad : \quad f \to \left(\frac{1}{s} D_s(sD_s) + \frac{1}{s^2} D_t^2\right) f$$

Now we can look at the individual terms:

$$\begin{aligned} \frac{1}{s} D_s(sD_s) &= \frac{1}{s}[\cos t D_x + \sin t D_y](s[\cos t D_x + \sin t D_y]) \\ &= \frac{1}{s^2}(xD_x + yD_y)(xD_x + yD_y) \\ &= \frac{1}{s^2} x(D_x + xD_x^2 + yD_xD_y) + \frac{1}{s^2} y(xD_xD_y + D_y + yD_y^2) \\ &= \frac{1}{s^2}\left(x(D_x + xD_x^2 + yD_xD_y) + y(xD_xD_y + D_y + yD_y^2)\right) \\ &= \frac{1}{s^2}\left(xD_x + yD_y + x^2D_x^2 + 2xyD_xD_y + y^2D_y^2\right) \end{aligned}$$

and from the second part,

$$\begin{aligned} \frac{1}{s^2} D_t^2 &= \frac{1}{s^2}[-s\sin t D_x + s\cos t D_y][-s\sin t D_x + s\cos t D_y] \\ &= \frac{1}{s^2}[-yD_x + xD_y][-yD_x + xD_y] \\ &= -\frac{1}{s^2} y[-yD_x^2 + D_y + xD_xD_y] + \frac{1}{s^2} x[-D_x - yD_xD_y + xD_y^2] \\ &= \frac{1}{s^2}\left(y^2D_x^2 - yD_y - xyD_xD_y - xD_x - xyD_xD_y + x^2D_y^2\right) \\ &= \frac{1}{s^2}\left(-xD_x - yD_y + y^2D_x^2 - 2xyD_xD_y + x^2D_y^2\right) \end{aligned}$$

Finally we add the two terms:

$$\begin{aligned} L &= \frac{1}{s^2}\left(xD_x + yD_y + x^2D_x^2 + 2xyD_xD_y + y^2D_y^2 - xD_x - yD_y + y^2D_x^2 - 2xyD_xD_y + x^2D_y^2\right) \\ &= \frac{1}{s^2}\left((x^2+y^2)D_x^2 + (x^2+y^2)D_y^2\right) \\ &= \frac{1}{s^2}\left(s^2D_x^2 + s^2D_y^2\right) = D_x^2 + D_y^2 = \nabla^2 \end{aligned}$$

so we have confirmed the form of ∇^2 just as we did before.