B3D Handout 12: The Integrating Factor Method.

This method is used to solve equations which are

- Ordinary differential equations
- Linear
- First-order

with any kind of coefficients.
A first order linear differential equation for $y(x)$ must be of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x).$$

If there is something multiplying the $\mathrm{d}y/\mathrm{d}x$ term, then divide the whole equation by this first.
Now suppose we calculate an **integrating factor**

$$I(x) = \exp\left(\int p(x)\,\mathrm{d}x\right).$$

Just this once, we won't bother about a constant of integration.
We multiply our equation by the integrating factor:

$$I(x)\frac{\mathrm{d}y}{\mathrm{d}x} + I(x)p(x)y = I(x)q(x).$$

and then observe that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(yI(x)\right) = \frac{\mathrm{d}y}{\mathrm{d}x}I(x) + y\frac{\mathrm{d}I}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x}I(x) + yp(x)I(x)$$

which is our left-hand-side. So we have the equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(yI(x)\right) = I(x)q(x)$$

which we can integrate (we hope):

$$yI(x) = \int I(x)q(x)\,\mathrm{d}x + C$$

$$y = \frac{1}{I(x)}\int I(x)q(x)\,\mathrm{d}x + \frac{C}{I(x)}.$$

This is the general solution of the equation.
If we have initial conditions, then the very **last** thing we do is determine the constant C.