B3D Handout 13: ODEs with Coefficients x^r

Suppose we are given a linear ordinary differential equation in which the coefficient of the rth derivative is a constant multiple of x^r:

$$\text{e.g.}\quad x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2x\frac{\mathrm{d}y}{\mathrm{d}x}-6y=3x^4.$$

Solving this is a two-stage process.

Complementary Function

The first step is to look for the complementary function (**CF**), which is the general solution of the *homogeneous equation*

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2x\frac{\mathrm{d}y}{\mathrm{d}x}-6y=0.$$

If we try a solution of the form $y=x^m$ we get

$$y=x^m \qquad \frac{\mathrm{d}y}{\mathrm{d}x}=mx^{m-1} \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=m(m-1)x^{m-2}$$

and if we put this back into the original equation we get

$$x^2m(m-1)x^{m-2}+2mxx^{m-1}-6x^m=0$$

$$(m^2+m-6)x^m=0 \qquad (m-2)(m+3)=0.$$

In this case we get two roots: $m_1=2$ and $m_2=-3$. This means we have found two functions that work as solutions to our differential equation:

$$y_1=x^{m_1}=x^2 \qquad \text{and} \qquad y_2=x^{m_2}=x^{-3}.$$

The *general solution* to the homogeneous equation is the general linear combination of these:

$$y=c_1x^2+c_2x^{-3}.$$

This works with an nth order ODE as long as the nth order polynomial for m has n different real roots.

Particular Integral

The particular integral (**PI**) is *any* solution of the whole equation.

We can only cope with one specific kind of RHS: a polynomial. For our example, the RHS is $3x^4$ so as long as the power on the right is **not part of the CF** we try a multiple of the right hand side:

$$y=Ax^4 \implies \frac{\mathrm{d}y}{\mathrm{d}x}=4Ax^3 \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=12Ax^2.$$

When we substitute this into the whole equation we get

$$x^2(12Ax^2)+2x(4Ax^3)-6(Ax^4)=3x^4.$$

$$12Ax^4+8Ax^4-6Ax^4=3x^4; \qquad 14A=3; \qquad A=\frac{3}{14}.$$

Our **PI** is

$$y_{\mathrm{PI}}=\frac{3}{14}x^4$$

so the general solution to the full equation is

$$y=c_1x^2+c_2x^{-3}+\frac{3}{14}x^4.$$

A couple of words of warning about this kind of equation:

- If the polynomial for the power m has a repeated root then we fail
- If the polynomial for the power m has complex roots then we fail
- If a power on the RHS matches a power in the CF then we fail.