B3D Handout 14: ODEs with Constant Coefficients

Suppose we are given a linear ordinary differential equation with constant coefficients:

$$\text{e.g.}\quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}+5\frac{\mathrm{d}y}{\mathrm{d}x}+6y=\sin x.$$

Solving this, like the ones with powers of x, is a two-stage process.

Complementary Function

The complementary function (**CF**) is the general solution of the homogeneous equation:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+5\frac{\mathrm{d}y}{\mathrm{d}x}+6y=0.$$

For this kind of equation we try a function $e^{\lambda x}$:

$$\lambda^2e^{\lambda x}+5\lambda e^{\lambda x}+6e^{\lambda x}=0 \qquad (\lambda^2+5\lambda+6)e^{\lambda x}=0 \qquad (\lambda+2)(\lambda+3)=0.$$

Here, we have roots $\lambda=-2$, $\lambda=-3$ so the CF is

$$y_{\mathrm{CF}}=c_1e^{-2x}+c_2e^{-3x}.$$

If we have a repeated root λ then we need

$$y_{\mathrm{CF}}=c_1e^{\lambda x}+c_2xe^{\lambda x}$$

and if a pair of roots are complex, $\lambda=a\pm ib$ then we use

$$y_{\mathrm{CF}}=e^{ax}(c_1 cosbx+c_2\sin bx).$$

Particular Integral

Again, we are looking for any solution to the whole equation. Typically we try something that looks like the right hand side. In our example the right hand side is $\sin x$ so we try:

$$y=A\sin x+B\cos x \qquad \frac{\mathrm{d}y}{\mathrm{d}x}=A\cos x-B\sin x \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-A\sin x-B\cos x$$

$$-A\sin x-B\cos x+5(A\cos x-B\sin x)+6(A\sin x+B\cos x)=\sin x.$$

$$5(A-B)\sin x+5(A+B)\cos x=\sin x \qquad -B=A=\frac{1}{10} \qquad y_{\mathrm{PI}}=\frac{1}{10}(\sin x-\cos x).$$

Here are a few right hand sides, with the standard first guess:

RHS	:	$e^{\lambda x}$	$\sin x$ or $\cos x$	x^2
PI	:	$Ae^{\lambda x}$	$A\sin x+B\cos x$	Ax^2+Bx+C

If the natural first guess contains the CF function, then we multiply the guess by x before we start. In general, if the first try fails, try multiplying by x and starting again.
The general solution is the sum of the CF and the PI: in our case,

$$y=c_1e^{-2x}+c_2e^{-3x}+\frac{1}{10}(\sin x-\cos x).$$