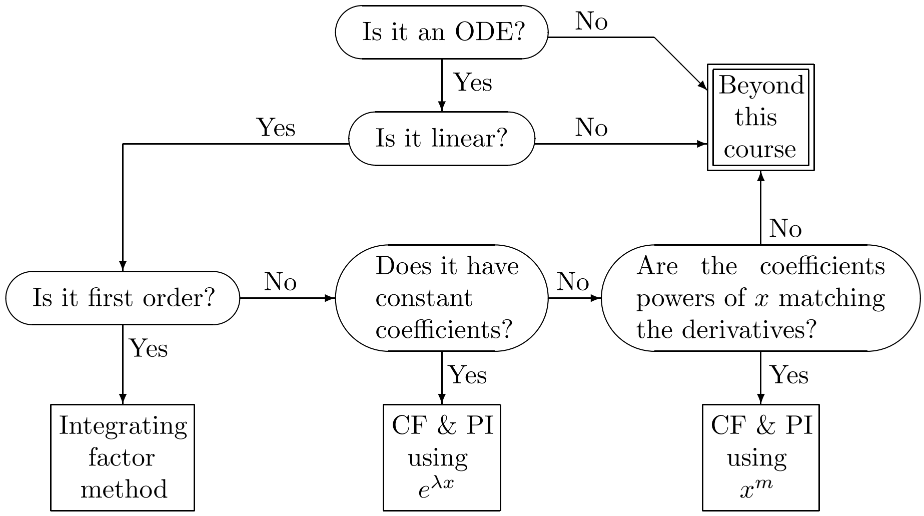
B3D Handout 15: Summary of differential equations



First-order linear ODEs

To solve an equation of the form $\dfrac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x),$ we calculate an integrating factor $I(x) = e^{\int p(x)\,\mathrm{d}x}$

and multiply by it: $\dfrac{\mathrm{d}y}{\mathrm{d}x}e^{\int p\,\mathrm{d}x} + p(x)ye^{\int p\,\mathrm{d}x} = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(ye^{\int p\,\mathrm{d}x}\right) = q(x)e^{\int p\,\mathrm{d}x}.$

We integrate both sides: $ye^{\int p\,\mathrm{d}x} = \displaystyle\int q(x)e^{\int p\,\mathrm{d}x}\,\mathrm{d}x + C \implies y = e^{-\int p\,\mathrm{d}x}\int q(x)e^{\int p\,\mathrm{d}x}\,\mathrm{d}x + Ce^{-\int p\,\mathrm{d}x}.$

Linear ODEs with constant coefficients

To solve an equation of the form $a\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\dfrac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x)$ we calculate the CF, by trying solutions of the form $y = e^{\lambda x}$ for the homogeneous equation (RHS=0). If there is a repeated root we use $e^{\lambda x}$ and $xe^{\lambda x}$:

$$y_{\mathrm{CF}} = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x} \qquad \text{or} \qquad y_{\mathrm{CF}} = c_1e^{\lambda x} + c_2xe^{\lambda x}.$$

We calculate the PI, which is **any** solution to the original equation, by a system of trial and error. In general we try something of the same form as $f(x)$; if this overlaps with the CF at all then we multiply by x. The general solution is $y = y_{\mathrm{CF}} + y_{\mathrm{PI}}$.

Linear ODEs with x^n-type coefficients

To solve an equation of the form $ax^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + bx\dfrac{\mathrm{d}y}{\mathrm{d}x} + cy = x^n$ we calculate the CF, by trying solutions of the form $y = x^m$ for the homogeneous equation. As long as the roots m are *real*, *different* and *not equal to* n this is OK. Then we use $y = Ax^n$ as the PI.