B3D Handout 16: Fourier series

If we have a **periodic** function $f(x)$ with period $2L$, that is,

$$f(x+2L) = f(x) \qquad \text{for all } x,$$

then the Fourier series for $f(x)$ is given by

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{\pi n x}{L}\right) + b_n \sin\left(\frac{\pi n x}{L}\right) \right\}.$$

Because of the *orthogonality relations*:

$$\begin{aligned}
\int_0^{2L} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) \mathrm{d}x &= \begin{cases} 0 & m \neq n \\ 2L & m = n = 0 \\ L & m = n \neq 0 \end{cases} \\
\int_0^{2L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) \mathrm{d}x &= \begin{cases} 0 & m \neq n \\ 0 & m = n = 0 \\ L & m = n \neq 0 \end{cases} \\
\int_0^{2L} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) \mathrm{d}x &= 0
\end{aligned}$$

if we multiply the $f(x)$ equation by one of the cos or sin functions and integrate we can show that the constants are given by

$$\begin{aligned}
a_n &= \frac{1}{L} \int_0^{2L} f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x \quad \text{for } n \geq 0 \\
b_n &= \frac{1}{L} \int_0^{2L} f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x \quad \text{for } n \geq 1
\end{aligned}$$

Even functions

If $f(x)$ is even, that is, $f(-x) = f(x)$, then the coefficients are

$$a_n = \frac{2}{L} \int_0^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x, \qquad b_n = 0.$$

Odd functions

If $f(x)$ is odd, that is, $f(-x) = -f(x)$, then the coefficients are

$$a_n = 0, \qquad b_n = \frac{2}{L} \int_0^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x.$$