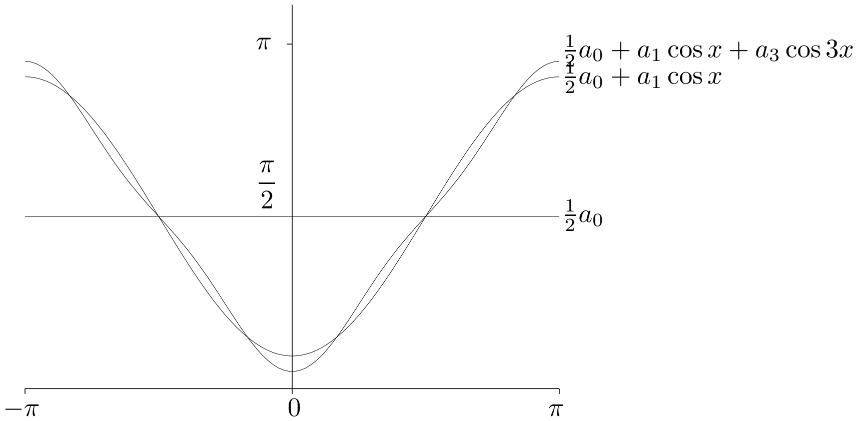
B3D Handout 17: Fourier Series Examples

Sawtooth function

$$f(x) = \begin{cases} -x & -\pi \le x \le 0 \\ x & 0 \le x \le \pi \end{cases} \quad \text{an even function.}$$

We calculated: $a_0 = \pi \qquad a_n = \dfrac{2}{\pi n^2}(\cos n\pi - 1) = \dfrac{2}{n^2\pi}\begin{cases} 0 & n \text{ even} \\ -2 & n \text{ odd} \end{cases} \qquad b_n = 0.$

The Fourier series begins: $$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right).$$



The coefficients in this case decrease fast, like $1/n^2$: this sort of decay occurs when f is *continuous*.

Square wave

$$f(x) = \begin{cases} -1 & -\pi < x \le 0 \\ 1 & 0 < x \le \pi \end{cases} \quad \text{an odd function.}$$

This function is periodic, but it is not continuous: there are jumps at 0, π, 2π, 3π, ...

We calculated: $a_n = 0 \qquad b_n = \dfrac{2}{n\pi}(1 - \cos n\pi) = \dfrac{2}{n\pi}\begin{cases} 2 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

and the Fourier series starts: $$f(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right).$$

Here the coefficients decrease slower, like $1/n$: this happens when f is discontinuous (has a jump).