B3D Handout 18: Handy Fourier Tricks

Here are just a couple of things we can do with the Fourier series: the standard Fourier series for a function with period $2L$ is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right\}.$$

1 Integration of a Fourier series

The Fourier series for $f(x)$ can be integrated term by term provided that $f(x)$ is piecewise continuous in the period $2L$ (i.e. only a finite number of jumps):

$$\int_\alpha^\beta f(x)\,\mathrm{d}x = \int_\alpha^\beta \frac{1}{2}a_0\,\mathrm{d}x + \sum_{n=1}^{\infty} \left\{ a_n \int_\alpha^\beta \cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x + b_n \int_\alpha^\beta \sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x \right\}.$$

2 Parseval's identity

We can multiply $f(x)$ by itself and integrate:

$$\begin{aligned}
\int_0^{2L} f(x)f(x)\,\mathrm{d}x &= \int_0^{2L} \left\{ \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right) \right\} f(x)\,\mathrm{d}x \\
&= \frac{a_0}{2}\int_0^{2L} f(x)\,\mathrm{d}x + \sum_{n=1}^{\infty} \left(a_n \int_0^{2L} \cos\frac{n\pi x}{L} f(x)\,\mathrm{d}x + b_n \int_0^{2L} \sin\frac{n\pi x}{L} f(x)\,\mathrm{d}x \right) \\
&= \frac{a_0}{2}La_0 + \sum_{n=1}^{\infty} (a_n L a_n + b_n L b_n) = L\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \right].
\end{aligned}$$

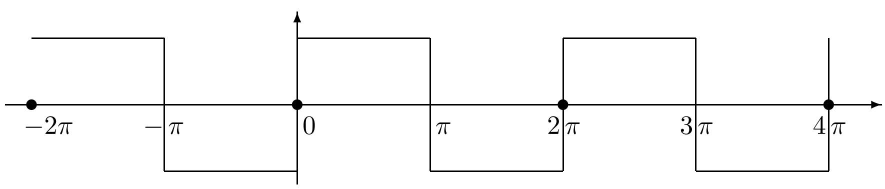
To put it another way,

$$\frac{1}{L}\int_0^{2L} f(x)f(x)\,\mathrm{d}x = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

This is **Parseval's identity**.

Example

Remember the square wave, of height 1 and period 2π:



The Fourier series for this function was

$$f(x) = \sum_1^{\infty} b_n \sin nx \qquad \text{with} \quad b_n = \begin{cases} \dfrac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

Parseval's identity gives

$$\frac{1}{\pi}\int_0^{2\pi} f^2(x)\,\mathrm{d}x = \sum_{n=1}^{\infty} b_n^2; \qquad 2 = \frac{16}{\pi^2}\left[1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots\right]$$

which tells us that

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots = \frac{\pi^2}{8}.$$