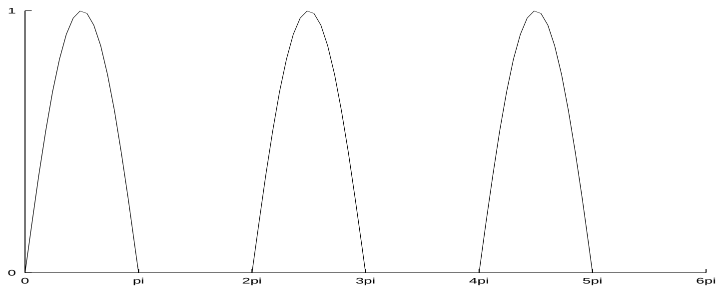
B3D Handout 19: Sheet 7 Question 2

We were looking at the "rectified wave" $f(x)$ with period 2π:

$$f(x) = \begin{cases} \sin x & \text{for } x \text{ between } 0 \text{ and } \pi \\ 0 & \text{for } x \text{ between } \pi \text{ and } 2\pi. \end{cases}$$

The solution (with an extra sketch) to your question:

(a) $f(x)$:



(b)

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\,\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi}\sin x\,\mathrm{d}x = \frac{1}{\pi}[-\cos x]_0^{\pi} = \frac{1}{\pi}[-\cos\pi + \cos 0] = \frac{2}{\pi} \\
a_1 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x\,\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi}\sin x\cos x\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{\pi}\sin 2x\,\mathrm{d}x \\
&= \frac{1}{2\pi}[-\tfrac{1}{2}\cos 2x]_0^{\pi} = \frac{1}{2\pi}[-\tfrac{1}{2}\cos 2\pi + \tfrac{1}{2}\cos 0] = 0 \\
b_1 &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin x\,\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi}\sin^2 x\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{\pi} 1-\cos 2x\,\mathrm{d}x \\
&= \frac{1}{2\pi}[x - \tfrac{1}{2}\sin 2x]_0^{\pi} = \frac{1}{2\pi}[\pi - \tfrac{1}{2}\sin 2\pi - 0 + \tfrac{1}{2}\sin 0] = \frac{1}{2} \\
a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi}\sin x\cos nx\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{\pi}(\sin(n+1)x - \sin(n-1)x)\,\mathrm{d}x \\
&= \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{(n+1)} + \frac{\cos(n-1)x}{(n-1)}\right]_0^{\pi} = \frac{1}{2\pi}\left[-\frac{(-1)^{n+1}}{(n+1)} + \frac{(-1)^{n-1}}{(n-1)} + \frac{1}{(n+1)} - \frac{1}{(n-1)}\right] \\
&= \frac{1}{2\pi}\left[\frac{(-1)^n+1}{(n+1)} - \frac{(-1)^n+1}{(n-1)}\right] = \begin{cases} 0 & n \text{ odd} \\ -2/[(n^2-1)\pi] & n \text{ even} \end{cases} \\
b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi}\sin x\sin nx\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{\pi}(\cos(n-1)x - \cos(n+1)x)\,\mathrm{d}x \\
&= \frac{1}{2\pi}[\frac{\sin(n-1)x}{(n-1)} - \frac{\sin(n+1)x}{(n+1)}]_0^{\pi} = 0
\end{aligned}$$

The series begins:

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{3\pi}\cos 2x$$