B3D Handout 2: Example Functions of Two Variables

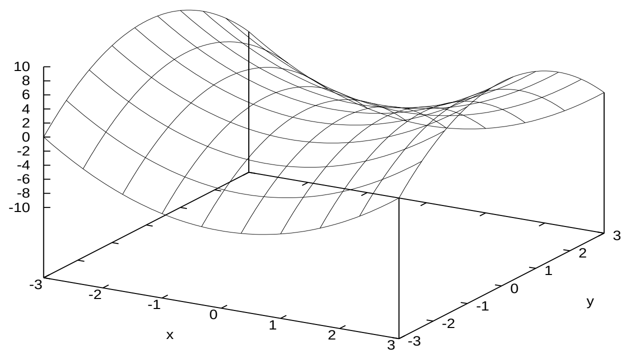
For a function of two variables, $f(x, y)$, consider (x, y) as defining a point P in the xy-plane. Let the value of $f(x, y)$ be taken as the length PP' drawn parallel to the z-axis (or the height of point P' above the plane). Then as P moves in the xy-plane, P' maps out a *surface* in space whose equation is $z = f(x, y)$.

Example: $f(x, y) = 6 - 2x - 3y$

The surface $z = 6 - 2x - 3y$, i.e. $2x + 3y + z = 6$, is a plane
which intersects the x-axis where $y = z = 0$, i.e. $x = 3$;
which intersects the y-axis where $x = z = 0$, i.e. $y = 2$;
which intersects the z-axis where $x = y = 0$, i.e. $z = 6$.

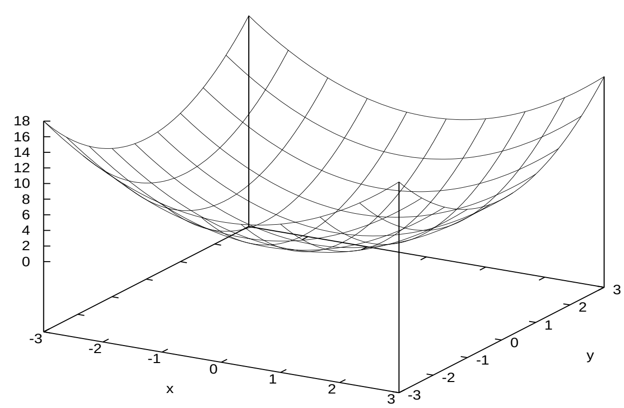
Example: $f(x, y) = x^2 - y^2$

In the plane $x = 0$, there is a *maximum* at $y = 0$; in the plane $y = 0$, there is a *minimum* at $x = 0$. The whole surface is shaped like a horse's saddle; and the picture shows a structure for which $(0, 0)$ is called a *saddle point*.



Example: $f(x, y) = x^2 + y^2$

The surface $z = x^2 + y^2$ may be drawn most easily by first converting into plane polar coordinates. Substituting $x = r\cos\theta$ and $y = r\sin\theta$ gives $z = r^2$. The surface is symmetric about the z-axis and its cross-section is a parabola. Thus the whole surface is a paraboloid (a bowl).



Another way to picture the same surface is to do as map-makers or weather forecasters do and draw *contour lines* (or *level curves*) – produces by taking a section, using a plane $z =$ const. and projecting it onto the xy-plane. For $z = x^2 + y^2$ as above, the contour lines are concentric circles.