# B3D Handout 20: Review of Linear Equations (B3C)

**Part (a): Notation.**

**Vectors**

$$\underline{v} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}, \quad \underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}.$$

If the vector has two components we call it a **two-dimensional vector**, three components make it a **three-dimensional vector** and so on.

**Linear combinations**

A linear combination of variables is an expression of the form

$$3x_1 + 2x_2, \qquad c_1 x + c_2 y + c_3 z$$

where $c_j$ $(j = 1, 2, 3)$ are constants.

In the same way we can write a linear combination of vectors:

$$4\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 3\begin{pmatrix} 6 \\ 4 \end{pmatrix}; \qquad \lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \mu \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

where $\lambda$, $\mu$ are constants.

**Linear equation** Examples of linear equations are:

$$x + y - 2z = 6 \qquad \text{(a plane in 3D space)}$$

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = A \qquad \text{with } a_1,\ a_2,\ a_3,\ a_4,\ A \text{ constant.}$$

**Set of linear equations** An example set of linear equations could be

$$\begin{array}{rcrcl} x & - & y & = & 2 \\ 3x & + & y & = & 4 \end{array}$$

or more generally,

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1N}x_N & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2N}x_N & = & b_2 \\ \vdots & & & & \ddots & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mN}x_N & = & b_m \end{array}$$

This is a set of $m$ linear equations in $N$ unknowns $x_1 \ldots x_N$, with constant coefficients $a_{11} \ldots a_{mN}$, $b_1 \ldots b_m$.

**Matrix notation** The sets of linear equations above can be written in matrix notation as

$$\begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \qquad \text{for the first, and} \qquad \underline{\underline{A}}\,\underline{x} = \underline{b} \qquad \text{for the second,}$$

$$\text{where} \qquad \underline{\underline{A}} = \begin{pmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mN} \end{pmatrix} \text{ is a matrix of constant coefficients,}$$

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \text{ is to be found, and } \underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \text{ is a constant vector.}$$

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**Part (b): Echelon Form.**

One way to solve a set of linear equations is by reduction to row-echelon form: using row operations:

- multiply a row by any constant (i.e. a number)
- interchange two rows
- add a multiple of one row to another

Row-echelon form means:

- any **all-zero** rows are at the bottom of the reduced matrix
- in a non-zero row, the first-from-left nonzero value is 1
- the first '1' in each row is to the right of the first '1' in the row above

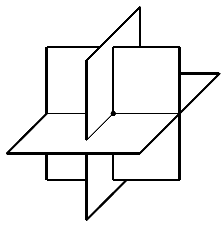
All the operations can be carried out on the augmented matrix

$$\left( \underline{\underline{A}} \mid \underline{b} \right).$$

Once complete, it is easy to find the solution by back-substitution.

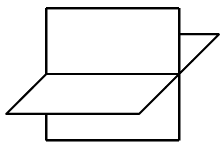
The **rank** of a matrix, $R$, is the number of non-zero rows when in echelon form. The ranks of the matrix and the augmented matrix determine the type of the solution:

- If we have $N$ variables and $N$ nonzero rows in echelon form, we always get a unique solution.
  In three dimensions this means geometrically that three independent planes meet at a point.



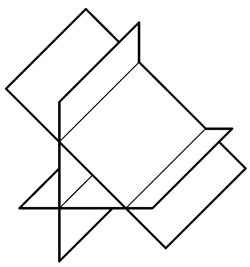
- If there is a zero row of the augmented matrix in echelon form, then the last variable can take any value and the general solution will have the form $\underline{x} = \underline{x}_1 + \lambda \underline{x}_2$ for known vectors $\underline{x}_1$ and $\underline{x}_2$ and variable $\lambda$. This happens when $R = N - 1$. $N$ is the number of variables in the problem.
- In general if the rank $R$ is less than the number of unknowns $N$, then there will be $N - R$ variables in the solution.

  In three dimensions with two independent rows this means geometrically that two independent planes intersect in a line.



- If the rank of the augmented matrix is greater than the rank of the original matrix, then there is no solution.

  Geometrically, this corresponds to a situation where three planes which are not independent do not intersect at all:



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**Part (c): Eigenvalues and Eigenvectors.**

For an $N \times N$ matrix $\underline{\underline{A}}$, if

$$\underline{\underline{A}}\,\underline{v} = \lambda \underline{v}$$

with $\underline{v} \neq \underline{0}$ then $\underline{v}$ is an **eigenvector** of $\underline{\underline{A}}$ with **eigenvalue** $\lambda$.
(Any multiple of $\underline{v}$ is also an eigenvector with eigenvalue $\lambda$: just pick some convenient form.)

$$\text{Now} \qquad \underline{\underline{A}}\,\underline{v} = \lambda \underline{v} \quad \Longrightarrow \quad (\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{v} = \underline{0}$$

which has non-zero solutions for $\underline{v}$ only if the determinant of matrix $\underline{\underline{A}} - \lambda \underline{\underline{I}}$ is zero.
The determinant $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}})$ is a polynomial in $\lambda$ of degree $N$, so there are at most $N$ different eigenvalues.
A useful fact (not proved here) is $\det(\underline{\underline{A}}) = \lambda_1 \lambda_2 \cdots \lambda_N$.

## Two properties of eigenvalues and eigenvectors

- Eigenvectors for different eigenvalues are linearly independent
- There may be multiple eigenvalues with the same value.

## Zero eigenvalues

It is possible for $\lambda = 0$ to be an eigenvalue.
However it is not possible for $\underline{v} = \underline{0}$ to be an eigenvector.

## Complex eigenvalues

Equally, $\lambda$ can be complex, in which case we expect the eigenvectors to be complex too. If the matrix $\underline{\underline{A}}$ is real then any complex eigenvalues will appear in complex conjugate pairs:

$$\lambda_1 = a + ib \qquad \lambda_2 = a - ib.$$