B3D Handout 21: The Gram-Schmidt Process

Remember, two vectors $\underline{v}_1$ and $\underline{v}_2$ are **orthogonal** if $\underline{v}_1 \cdot \underline{v}_2 = 0$.

We are given a set of *linearly independent* vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_N$. We want to create a set of *mutually orthogonal* vectors $\underline{g}_1, \ldots, \underline{g}_N$ each of which is a linear combination of the $\underline{v}$ vectors.

$$
\begin{aligned}
\text{Put } \underline{g}_1 &= \underline{v}_1 \\
\text{Then } \underline{g}_2 &= \underline{v}_2 - (\underline{v}_2 \cdot \hat{\underline{g}}_1)\hat{\underline{g}}_1 \quad = \underline{v}_2 - \frac{(\underline{v}_2 \cdot \underline{v}_1)}{(\underline{v}_1 \cdot \underline{v}_1)}\underline{v}_1 \text{ as before} \\
\underline{g}_3 &= \underline{v}_3 - (\underline{v}_3 \cdot \hat{\underline{g}}_1)\hat{\underline{g}}_1 - (\underline{v}_3 \cdot \hat{\underline{g}}_2)\hat{\underline{g}}_2 \\
&= \underline{v}_3 - \frac{(\underline{v}_3 \cdot \underline{g}_1)}{(\underline{g}_1 \cdot \underline{g}_1)}\underline{g}_1 - \frac{(\underline{v}_3 \cdot \underline{g}_2)}{(\underline{g}_2 \cdot \underline{g}_2)}\underline{g}_2 \\
\underline{g}_4 &= \underline{v}_4 - (\underline{v}_4 \cdot \hat{\underline{g}}_1)\hat{\underline{g}}_1 - (\underline{v}_4 \cdot \hat{\underline{g}}_2)\hat{\underline{g}}_2 - (\underline{v}_4 \cdot \hat{\underline{g}}_3)\hat{\underline{g}}_3
\end{aligned}
$$

and so on.

Let us check orthogonality (one example):

$$\underline{g}_3 \cdot \underline{g}_2 = \underline{v}_3 \cdot \underline{g}_2 - (\underline{v}_3 \cdot \hat{\underline{g}}_1)\hat{\underline{g}}_1 \cdot \underline{g}_2 - (\underline{v}_3 \cdot \hat{\underline{g}}_2)\hat{\underline{g}}_2 \cdot \underline{g}_2 = \underline{v}_3 \cdot \underline{g}_2 - 0 - (\underline{v}_3 \cdot \hat{\underline{g}}_2)|\underline{g}_2| = 0.$$

Example

$$\underline{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \underline{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

$$\underline{g}_1 = \underline{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}. \qquad \underline{g}_1 \cdot \underline{g}_1 = \qquad \underline{v}_2 \cdot \underline{g}_1 =$$

$$\underline{g}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - - \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} \\ \\ \end{pmatrix}.$$

$$\underline{g}_2 \cdot \underline{g}_2 = \qquad \underline{v}_3 \cdot \underline{g}_1 = \quad . \quad \underline{v}_3 \cdot \underline{g}_2 = \quad .$$

$$\underline{g}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \quad \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \quad \begin{pmatrix} \\ \\ \end{pmatrix} =$$

Note we can choose any multiple of these calculated vectors: so let us have

$$\underline{g}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \underline{g}_2 = \begin{pmatrix} \\ \\ \end{pmatrix}, \quad \underline{g}_3 = \begin{pmatrix} \\ \\ \end{pmatrix}.$$

Check orthogonality:

$$
\begin{aligned}
\underline{g}_1 \cdot \underline{g}_2 &= 0 \\
\underline{g}_1 \cdot \underline{g}_3 &= 0 \\
\underline{g}_2 \cdot \underline{g}_3 &= 0.
\end{aligned}
$$

Summary of Gram-Schmidt Process

$$
\begin{aligned}
\underline{g}_1 &= \underline{v}_1 \\
\underline{g}_k &= \underline{v}_k - \sum_{i=1}^{k-1} \frac{(\underline{v}_k \cdot \underline{g}_i)}{(\underline{g}_i \cdot \underline{g}_i)}\underline{g}_i \quad \text{for } k = 2, \cdots, N \\
&= \underline{v}_k - \sum_{i=1}^{k-1} (\underline{v}_k \cdot \hat{\underline{g}}_i)\hat{\underline{g}}_i.
\end{aligned}
$$