B3D Handout 22: Linear ODEs

Revision

If we are solving a linear first-order Ordinary Differential Equation (ODE) with constant coefficients, it will be of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + b, \quad \text{which we also write as} \quad \dot{x} = ax + b.$$

We look first at the **homogeneous** equation $\dot{x} = ax$ and try solutions of the form $x = e^{\lambda t}$, which gives

$$\lambda e^{\lambda t} = a e^{\lambda t} \implies \lambda = a.$$

The general solution to this equation is

$$x = ce^{at}.$$

Then we note that a **particular solution** of our full equation is $x = b$, and add this on, giving the general solution

$$x = ce^{at} + b.$$

System of equations and matrix form

Now let us look at a pair of **coupled** ODEs: this means that the equation for x has y in it and *vice versa*:

$$\begin{aligned} \dot{x} &= \alpha_1 x + \alpha_2 y \\ \dot{y} &= \alpha_3 x + \alpha_4 y \end{aligned} \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

If we put $\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, then $\underline{\dot{x}} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \underline{x} = \underline{\underline{A}}\,\underline{x}$.

Now let us try a solution of the form

$$\underline{x} = \underline{v} e^{\lambda t}.$$

This gives

$$\underline{\dot{x}} = \lambda \underline{v} e^{\lambda t} \implies \underline{\underline{A}}\,\underline{v} e^{\lambda t} = \lambda \underline{v} e^{\lambda t},$$

in other words, $\underline{v}$ must be an eigenvector of $\underline{\underline{A}}$ with eigenvalue λ.

Suppose we have found the two eigenvalues λ_1 and λ_2, with eigenvectors $\underline{v}_1$ and $\underline{v}_2$. Then there are two independent **modes** of solution:

$$\underline{x}_1 = \underline{v}_1 e^{\lambda_1 t}; \qquad \underline{x}_2 = \underline{v}_2 e^{\lambda_2 t}$$

and the general solution can be built from them:

$$\underline{x} = c_1 \underline{x}_1 + c_2 \underline{x}_2 \text{ for some constants } c_1,\, c_2.$$

Constant terms in the governing equation

What if the system has, as well as the linear terms, some constants?

$$\underline{\dot{x}} = \underline{\underline{A}}\,\underline{x} + \underline{b} \quad \text{where } \underline{b} \text{ is a constant vector.}$$

This has a particular (constant) solution $\underline{p}$ with

$$\underline{0} = \underline{\underline{A}}\,\underline{p} + \underline{b} \implies \underline{\underline{A}}\,\underline{p} = -\underline{b}$$

We can solve for $\underline{p}$ using row-echelon form as usual, and then the general solution to the governing equation is

$$\underline{x} = c_1 \underline{x}_1 + c_2 \underline{x}_2 + \underline{p}.$$