B3D Handout 23: More Linear ODEs

Systems with insufficient eigenvectors

Beware: sometimes, if there is a repeated eigenvalue, there may not be N eigenvectors to the N-equation system.

For example, consider a two-by-two system with one repeated root, eigenvalue λ, and only one eigenvector, $\underline{v}$. This gives us one solution:

$$\underline{x} = \underline{v}e^{\lambda t}.$$

To find another solution, we try the form

$$\begin{aligned}
\underline{x} &= (t\underline{v} + \underline{w})e^{\lambda t} \\
\dot{\underline{x}} &= \underline{v}e^{\lambda t} + \lambda(t\underline{v} + \underline{w})e^{\lambda t} = (t\lambda\underline{v} + \underline{v} + \lambda\underline{w})e^{\lambda t} \\
\underline{\underline{A}}\,\underline{x} &= \underline{\underline{A}}(t\underline{v} + \underline{w})e^{\lambda t} = (t\underline{\underline{A}}\,\underline{v} + \underline{\underline{A}}\,\underline{w})e^{\lambda t} = (t\lambda\underline{v} + \underline{\underline{A}}\,\underline{w})e^{\lambda t}
\end{aligned}$$

so this form satisfies $\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$ if

$$\underline{\underline{A}}\,\underline{w} = \underline{v} + \lambda\underline{w} \implies (\underline{\underline{A}} - \lambda\underline{\underline{I}})\underline{w} = \underline{v}$$

so $\underline{w}$ is the **generalised eigenvector** associated with λ.

Now we have two independent solutions:

$$\underline{x}_1 = \underline{v}e^{\lambda t} \qquad \underline{x}_2 = [t\underline{v} + \underline{w}]\,e^{\lambda t}$$

giving a general solution

$$\underline{x} = \alpha\underline{v}e^{\lambda t} + \beta\,[t\underline{v} + \underline{w}]\,e^{\lambda t},$$

where α and β are scalar constants that can be found from the initial conditions.

Determining constants from initial conditions

Suppose we were also told that $\underline{x} = \underline{x}_0$ when $t = 0$.

To find the constant vector $\underline{c}$, we must sort out the particular solution $\underline{p}$ **first**, and **then** apply the initial conditions.

Given the initial condition $\underline{x}_0$ at t_0,

$$c_1\underline{x}_1(t_0) + c_2\underline{x}_2(t_0) + \underline{p} = \underline{x}_0,$$

which is yet another linear equation (or set of equations) to solve for the constants c_1 and c_2. Usually we will solve this using row-echelon form.