B3D Handout 24: Matrix Diagonalisation

If an $N \times N$ matrix $\underline{\underline{A}}$ has N linearly independent eigenvectors $\underline{v}_n$, put

$$\underline{\underline{V}} = \begin{pmatrix} \underline{v}_1 & \cdots & \underline{v}_N \end{pmatrix}.$$

Then

$$\underline{\underline{A}}\,\underline{\underline{V}} = \begin{pmatrix} \lambda_1 \underline{v}_1 & \cdots & \lambda_N \underline{v}_N \end{pmatrix} = \underline{\underline{V}} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_N \end{pmatrix} = \underline{\underline{V}}\,\underline{\underline{\Lambda}}.$$

[Note that $\underline{\underline{V}}\,\underline{\underline{\Lambda}} \neq \underline{\underline{\Lambda}}\,\underline{\underline{V}}$; the order of multiplication of matrices is important.]
As the vectors $\underline{v}_n$ are linearly independent, $|\underline{\underline{V}}| \neq 0$ and we can invert $\underline{\underline{V}}$ to form $\underline{\underline{V}}^{-1}$. Then

$$\underline{\underline{V}}^{-1}\underline{\underline{A}}\,\underline{\underline{V}} = \underline{\underline{V}}^{-1}\underline{\underline{V}}\,\underline{\underline{\Lambda}} = \underline{\underline{\Lambda}}.$$

Expression for $\underline{\underline{A}}$

Since $\underline{\underline{A}}\,\underline{\underline{V}} = \underline{\underline{V}}\,\underline{\underline{\Lambda}}$, we can multiply on the right by $\underline{\underline{V}}^{-1}$ to have $\underline{\underline{A}} = \underline{\underline{V}}\,\underline{\underline{\Lambda}}\,\underline{\underline{V}}^{-1}$.

Summary of the method

- Find eigenvectors and eigenvalues: this only works if we have N eigenvectors
- $\underline{\underline{V}} = \begin{pmatrix} \underline{v}_1 & \cdots & \underline{v}_N \end{pmatrix}$
- $\underline{\underline{\Lambda}} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_N \end{pmatrix}$.
- Calculate $\underline{\underline{V}}^{-1}$
- Check $\underline{\underline{V}}^{-1}\underline{\underline{A}}\,\underline{\underline{V}} = \underline{\underline{\Lambda}}$.

Common special case: $\underline{\underline{A}}$ symmetric

If $\underline{\underline{A}}$ is **symmetric**, that is, $\underline{\underline{A}}^\top = \underline{\underline{A}}$, then

- its eigenvalues are real.
- the eigenvectors $\underline{v}_i$ and $\underline{v}_j$ are **orthogonal** if $\lambda_i \neq \lambda_j$.
- the eigenvectors for $\lambda_i = \lambda_j$ can be made orthogonal if necessary.

We can always choose eigenvectors of length 1, or **normalise**, so we get an orthonormal set:

$$\underline{v}_i \cdot \underline{v}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad \text{and} \quad \underline{v}_i^\top \underline{v}_j = \underline{v}_i \cdot \underline{v}_j.$$

Then

$$\underline{\underline{V}}^\top \underline{\underline{V}} = \begin{pmatrix} \underline{v}_1^\top \\ \vdots \\ \underline{v}_N^\top \end{pmatrix} \begin{pmatrix} \underline{v}_1 & \cdots & \underline{v}_N \end{pmatrix} = \begin{pmatrix} \underline{v}_1^\top \underline{v}_1 & \cdots & \underline{v}_1^\top \underline{v}_N \\ \vdots & \ddots & \vdots \\ \underline{v}_N^\top \underline{v}_1 & \cdots & \underline{v}_N^\top \underline{v}_N \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is the identity matrix, so

$$\boxed{\underline{\underline{V}}^\top = \underline{\underline{V}}^{-1}.}$$