B3D Handout 25: Linear ODEs and diagonalisation

Suppose our ODE system has a matrix form in which the matrix has a complete set of N linearly independent eigenvectors, and so can be reduced to the diagonal matrix $\underline{\underline{\Lambda}}$:

$$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} + \underline{b} \qquad \underline{\underline{A}}\,\underline{\underline{V}} = \underline{\underline{V}}\,\underline{\underline{\Lambda}} \qquad \underline{\underline{V}} = \begin{pmatrix} \underline{v}_1 & \cdots & \underline{v}_N \end{pmatrix}.$$

Now put $\underline{x} = \underline{\underline{V}}\,\underline{X}$. This gives $\dot{\underline{x}} = \underline{\underline{V}}\,\dot{\underline{X}}$ and so

$$\underline{\underline{V}}\,\dot{\underline{X}} = \underline{\underline{A}}\,\underline{\underline{V}}\,\underline{X} + \underline{b} \implies \dot{\underline{X}} = \underline{\underline{V}}^{-1}\underline{\underline{A}}\,\underline{\underline{V}}\,\underline{X} + \underline{\underline{V}}^{-1}\underline{b} \implies \dot{\underline{X}} = \underline{\underline{\Lambda}}\,\underline{X} + \underline{\underline{V}}^{-1}\underline{b}$$

This is now an uncoupled system: e.g. in two dimensions

$$\underline{\underline{V}} = \begin{pmatrix} \underline{v}_1 & \underline{v}_2 \end{pmatrix} \qquad \underline{\underline{\Lambda}} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \underline{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \underline{X} = \begin{pmatrix} X \\ Y \end{pmatrix} \qquad \underline{\underline{V}}^{-1}\underline{b} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

then

$$\begin{aligned} \dot{X} &= \lambda_1 X + \alpha \\ \dot{Y} &= \lambda_2 Y + \beta \end{aligned}$$

X does not depend on Y and *vice versa*. We can easily solve each equation independently.

At this point you would work with the equations specific to your example, but in general terms the solution (for a two-equation system) is

$$\begin{aligned} X &= c_1 e^{\lambda_1 t} - \alpha/\lambda_1 \\ Y &= c_2 e^{\lambda_2 t} - \beta/\lambda_2 \end{aligned}$$

and then we can use $\underline{\underline{V}}$ again to get the final result:

$$\underline{x} = \underline{\underline{V}}\,\underline{X} = \begin{pmatrix} \underline{v}_1 & \underline{v}_2 \end{pmatrix} \begin{pmatrix} c_1 e^{\lambda_1 t} - \alpha/\lambda_1 \\ c_2 e^{\lambda_2 t} - \beta/\lambda_2 \end{pmatrix} = c_1 \underline{v}_1 e^{\lambda_1 t} + c_2 \underline{v}_2 e^{\lambda_2 t} - \frac{\alpha}{\lambda_1}\underline{v}_1 - \frac{\beta}{\lambda_2}\underline{v}_2.$$