B3D Handout 3: Review of Partial Differentiation

For a function f that depends on **several** variables x, y, $\ldots$ we can differentiate with respect to each of these variables, keeping the others constant. This process is called *partial differentiation*.

The process is just like ordinary differentiation except that every time we see a variable other than the one we're considering, we treat it like a constant.

Example: If $f(x, y) = x^4y^2 - x^2y^6$ then

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 4x^3y^2 - 2xy^6 \\
\frac{\partial f}{\partial y} &= 2x^4y - 6x^2y^5 \\
\frac{\partial^2 f}{\partial x^2} &= 12x^2y^2 - 2y^6 \\
\frac{\partial^2 f}{\partial y \partial x} &= 8x^3y - 12xy^5 \\
\frac{\partial^2 f}{\partial y^2} &= 2x^4 - 30x^2y^4 \\
\frac{\partial^2 f}{\partial x \partial y} &= 8x^3y - 12xy^5
\end{aligned}$$

The Mixed Derivatives Theorem

The Mixed Derivative Theorem states that if f_{xy} and f_{yx} are continuous then $f_{xy} = f_{yx}$.

Thus to calculate a mixed derivative we can calculate in either order.

For third-order derivatives the mixed derivatives theorem gives $f_{xxy} = f_{xyx} = f_{yxx}$.

Physical meaning of the partial derivative of a function of two variables

Remember that we can think of a function of two variables, $f(x, y)$, as representing a surface or landscape

$$z = f(x, y).$$

- For a function $f(x)$, the *ordinary* derivative $\mathrm{d}f/\mathrm{d}x$ gives the slope of the tangent to the curve $y = f(x)$ at any point P.
- For a function $f(x, y)$, the *partial* derivative $\partial f/\partial x$ is evaluated holding y constant, and so gives the slope of the tangent to the surface in a plane $y =$ constant.
- In the same way, the partial derivative $\partial f/\partial y$ is taken keeping x constant so it represents the slope of the surface $z = f(x, y)$ along the y-direction, in a plane of constant x.