B3D Handout 5: Spherical polars example

Question: Express in spherical polar coordinates the solid T that is bounded above by the cone $z^2 = x^2 + y^2$, below by the xy-plane, and on the sides by the hemisphere $z = (4 - x^2 - y^2)^{1/2}$.
Solution: The solid is defined by the following inequalities:

$$z^2 \leq x^2 + y^2 \qquad z \geq 0 \qquad x^2 + y^2 + z^2 \leq 4.$$

Substituting the definitions of x, y and z in terms of ρ, θ and ϕ gives:

$$\rho^2 \cos^2\theta \leq \rho^2 \sin^2\theta \cos^2\phi + \rho^2 \sin^2\theta \sin^2\phi, \qquad \rho\cos\theta \geq 0$$

$$\rho^2 \sin^2\theta \cos^2\phi + \rho^2 \sin^2\theta \sin^2\phi + \rho^2\cos^2\theta \leq 4$$

so

$$\rho^2\cos^2\theta \leq \rho^2\sin^2\theta, \qquad \rho\cos\theta \geq 0, \qquad \rho^2 \leq 4.$$

and we can use the fact that $\rho \geq 0$ and $\sin\theta \geq 0$ to deduce

$$\rho \leq 2 \qquad \cos\theta \geq 0 \qquad 1 \leq \tan\theta.$$

Given that $0 \leq \theta \leq \pi$, this reduces to:

$$\begin{aligned} 0 \leq \ &\rho \ \leq 2 \\ \pi/4 \leq \ &\theta \ \leq \pi/2. \end{aligned}$$

In this case, where there is no information about ϕ contained in our limits, we use the whole permitted range:

$$0 \leq \phi < 2\pi.$$