B3D Handout 6: Critical points of $f(x, y)$

Definition: For a function of two variables, $f(x, y)$, a *critical point* is defined to be a point at which both of the first partial derivatives are zero:

$$\frac{\partial f}{\partial x} = 0, \qquad \frac{\partial f}{\partial y} = 0.$$

We can **classify** a critical point using two key quantities:

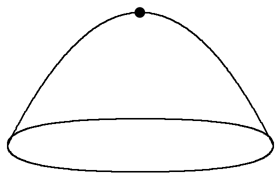
- f_{xx}, the second partial derivative of f with respect to x, and
- $H = f_{xx} f_{yy} - f_{xy}^2$, the **Hessian**.

Degenerate or non-degenerate?

- If the Hessian is zero, then our critical point is **degenerate**.
- For a **non-degenerate** critical point, for which the Hessian is nonzero, there are three possible types of behaviour.

Maximum

This happens if the Hessian is positive **and** f_{xx} is negative:

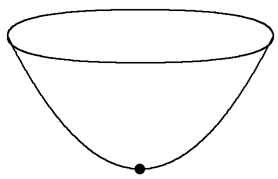


Sufficient conditions for a **maximum** at a critical point are that $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at that point.

The function decreases as you move away from the critical point in any direction.

Minimum

This happens when the Hessian is positive and so is f_{xx}:

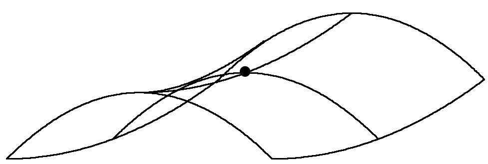


Sufficient conditions for a **minimum** at a critical point are that $f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at that point.

The function increases as you move away from the critical point in any direction.

Saddle Point

This happens if the Hessian is negative:



Sufficient condition for a **saddle point** is that $f_{xx} f_{yy} - f_{xy}^2 < 0$ at that point.

As you move away from the critical point, the function may increase or decrease depending on which direction you choose.