B3D Handout 8: Gradient Properties

The gradient vector in two dimensions

A function of two variables $f(x, y)$ can be used to represent a surface using

$$z = f(x, y).$$

The gradient of this function is

$$\underline{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right).$$

The key properties of the gradient vector in two dimensions are:

- **Property 1.** At any point, $\underline{\nabla} f$ points in the direction in which f is increasing most rapidly: i.e. $\underline{\nabla} f$ points uphill. Its magnitude $|\underline{\nabla} f|$ gives the slope in this steepest direction.
- **Property 2.** At any point, $\underline{\nabla} f$ is perpendicular to the contour line $f =$ const. through that point.

The gradient vector in three dimensions

A function of three variables $f(x, y, z)$ can still be used to represent a surface: we use

$$f(x, y, z) = A$$

for some constant A.

The gradient vector in three dimensions is:

$$\underline{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right).$$

The key properties in three dimensions have an exact analogy with the ones above for a function of two variables.

- **Property 1.** $\underline{\nabla} f$ points in the direction in which f increases fastest, and its magnitude gives the rate of change of f in that direction.
- **Property 2.** $\underline{\nabla} f$ is perpendicular to the surface $f =$ constant.

The second property can be used to find a vector perpendicular to any surface when we want one: just write the surface in the form

$$f(x, y, z) = \text{constant}$$

and then take the gradient of f.