B3D Handout 9: Grad, div, curl and the Laplacian ∇^2

Grad

We have already met the gradient vector in three dimensions:

$$\underline{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).$$

Divergence, div

This is a scalar, that is, it just produces a number.

$$div(\underline{q}) = \underline{\nabla} \cdot \underline{q} = \frac{\partial q_1}{\partial x} + \frac{\partial q_2}{\partial y} + \frac{\partial q_3}{\partial z}.$$

Curl

The vector operator curl is based on the cross product:

$$\underline{curl}(\underline{q}) = \underline{\nabla} \times \underline{q} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ q_1 & q_2 & q_3 \end{vmatrix}.$$

Laplacian

The Laplacian is the divergence of the gradient vector:

$$\nabla^2 f = \underline{\nabla} \cdot \underline{\nabla} f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

Polar Coordinates

None of these are simple in polar coordinates.
These are the standard forms for the Laplacian in the common polar coordinate systems:

$$\begin{aligned}
\nabla^2 f &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} \text{ in plane polar coordinates} \\
\nabla^2 f &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \text{ in cylindrical polar coordinates} \\
\nabla^2 f &= \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} \text{ in spherical polar coordinates}
\end{aligned}$$

The divergence in plane polars, for a vector function

$$\underline{q} = q^{(r)}\underline{e}_r + q^{(\theta)}\underline{e}_\theta$$

is given by:

$$\underline{\nabla} \cdot \underline{q} = \frac{1}{r}\frac{\partial}{\partial r}(rq^{(r)}) + \frac{1}{r}\frac{\partial q^{(\theta)}}{\partial \theta}.$$

The vectors $\underline{e}_r$ and $\underline{e}_\theta$ are unit vectors in the r and θ directions respectively; one of the reasons everything is more complicated with polars is that these unit vectors depend on position.