Mathematics B3D

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1 Functions of Several Variables

1.1 Introduction

Since we live in a three-dimensional world, in applied mathematics we are interested in functions which can vary with any of the three space variables x, y, z and also with time t. For instance, if the function f represents the temperature in this room, then f depends on the location (x, y, z) at which it is measured and also on the time t when it is measured, so f is a function of the independent variables x, y, z and t, i.e. $f(x, y, z, t)$.

1.2 Geometric Interpretation

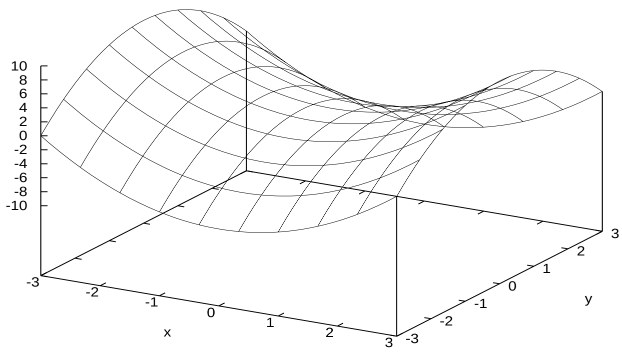
For a function of two variables, $f(x, y)$, consider (x, y) as defining a point P in the xy-plane. Let the value of $f(x, y)$ be taken as the length PP' drawn parallel to the z-axis (or the height of point P' above the plane). Then as P moves in the xy-plane, P' maps out a *surface* in space whose equation is $z = f(x, y)$.

Example: $f(x, y) = 6 - 2x - 3y$

The surface $z = 6 - 2x - 3y$, i.e. $2x + 3y + z = 6$, is a plane
which intersects the x-axis where $y = z = 0$, i.e. $x = 3$;
which intersects the y-axis where $x = z = 0$, i.e. $y = 2$;
which intersects the z-axis where $x = y = 0$, i.e. $z = 6$.

Example: $f(x, y) = x^2 - y^2$

In the plane $x = 0$, there is a *maximum* at $y = 0$; in the plane $y = 0$, there is a *minimum* at $x = 0$. The whole surface is shaped like a horse's saddle; and the picture shows a structure for which $(0, 0)$ is called a *saddle point*.

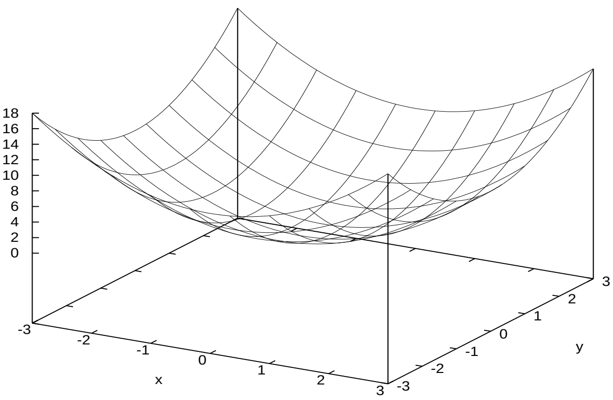


1.2.1 Plane polar coordinates

Since the variables x and y respresent a point in the plane, we can express that point in plane polar coordinates simply by substituting the definitions:

$$x = r\cos\theta \quad y = r\sin\theta.$$

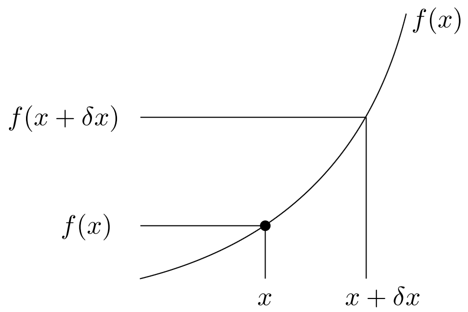
Example: $f(x,y) = x^2 + y^2$
The surface $z = x^2 + y^2$ may be drawn most easily by first converting into plane polar coordinates. Substituting $x = r\cos\theta$ and $y = r\sin\theta$ gives $z = r^2$. The surface is symmetric about the z-axis and its cross-section is a parabola. [Check with the original function and the plane $y = 0$.] Thus the whole surface is a paraboloid (a bowl).



Another way to picture the same surface is to do as map-makers or weather forecasters do and draw *contour lines* (or *level curves*) – produces by taking a section, using a plane $z =$ const. and projecting it onto the xy-plane. For $z = x^2 + y^2$ as above, the contour lines are concentric circles.

1.3 Partial Differentiation

Remember the derivative(!):



For a function $f(x)$ that depends on a single variable, x, the **ordinary derivative** is

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

and gives the slope of the curve $y = f(x)$ at the point x.
Example: $f(x) = 3x^4 + \sin x$

$$\frac{\mathrm{d}f}{\mathrm{d}x} = 12x^3 + \cos x.$$

For a function f that depends on **several** variables x, y, ... we can differentiate with respect to each of these variables, keeping the others constant. This process is called *partial differentiation* (*partial derivatives*).
Example: $f(x, y) = yx^4 + \sin x$
We treat y as a constant as we did 3 for the $f(x)$ above, and have

$$\frac{\partial f}{\partial x} = 4yx^3 + \cos x.$$

The formal definition (for a function of x and y) is

$$\frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}$$

and it gives the slope of a slice y =constant of the surface $z = f(x, y)$.
To find a **partial derivative** we hold all but one of the independent variables constant and differentiate with respect to that one variable using the ordinary rules for one-variable calculus.
Notation:
The partial derivative of f with respect to x is denoted by $\partial f/\partial x$ or by f_x.
Example: Calculate the partial derivatives of the functions:

(a) $f(x, y) = x^2 + 2xy^2 + y^3$;

(b) $f(x, y, z) = xz + e^{yz} + \sin(xy)$.

Solution:

(a) Holding y constant gives $\partial f/\partial x = 2x + 2y^2 + 0$.
Holding x constant gives $\partial f/\partial y = 0 + 4xy + 3y^2$.

(b) Holding both y and z constant gives $f_x = z + 0 + y\cos(xy)$.
Holding both x and z constant gives $f_y = 0 + ze^{yz} + x\cos(xy)$.
Holding both x and y constant gives $f_z = x + ye^{yz} + 0$.

1.3.1 Second-order partial derivatives

For $f(x,y)$ we can form $\partial f/\partial x$ and $\partial f/\partial y$. Each of these can then be differentiated again with respect to x or y to form the **second-order derivatives**

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx};$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy};$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx};$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}.$$

f_{xy} and f_{yx} are called **mixed derivatives**.

Example: If $f(x,y) = x^4y^2 - x^2y^6$ then

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 4x^3y^2 - 2xy^6 \\
\frac{\partial f}{\partial y} &= 2x^4y - 6x^2y^5 \\
\frac{\partial^2 f}{\partial x^2} &= 12x^2y^2 - 2y^6 \\
\frac{\partial^2 f}{\partial y \partial x} &= 8x^3y - 12xy^5 \\
\frac{\partial^2 f}{\partial y^2} &= 2x^4 - 30x^2y^4 \\
\frac{\partial^2 f}{\partial x \partial y} &= 8x^3y - 12xy^5
\end{aligned}$$

Note that in this example $f_{xy} = f_{yx}$ and this is true in general.

1.3.2 The Mixed Derivatives Theorem

The Mixed Derivative Theorem states that if f_{xy} and f_{yx} are continuous then $f_{xy} = f_{yx}$.

Thus to calculate a mixed derivative we can calculate in either order.

[Think about calculating $\partial/\partial x\,(\partial f/\partial y)$ if $f(x,y) = xy + 1/(\sin(y^2) + e^y)$.]

For third-order derivatives the mixed derivatives theorem gives $f_{xxy} = f_{xyx} = f_{yxx}$.

1.3.3 Partial differential equations

Just as we can use ordinary derivatives to write down a **differential equation** for a function we don't know:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + 3x\frac{\mathrm{d}f}{\mathrm{d}x} + f(x) = 0,$$

for many real-world physical situations, we are working in three-dimensional space or four-dimensional space-time, so the equations we need will be **partial differential equations**.

Quantum mechanics The wave-function describing how electrons behave, $\psi(x, y, z, t)$, is governed by **Schrödinger's equation**:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) + V(x, y, z)\psi.$$

Heat transfer The heat equation for the way the temperature changes in a conducting solid is

$$\frac{\partial T}{\partial t} = \kappa\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right).$$

All of the partial differential equations above will work better in a vector form: we will see how this sort of shorthand works later on.

1.4 The Chain Rule

Remember the chain rule for ordinary differentiation:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}.$$

This applies when f is a function of x and x depends on t.

We now generalise this result for a function of several variables. For $f(x, y)$, suppose x and y depend on t. The *chain rule* for a function of two variables is:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \left(\frac{\partial f}{\partial x}\right)\frac{\mathrm{d}x}{\mathrm{d}t} + \left(\frac{\partial f}{\partial y}\right)\frac{\mathrm{d}y}{\mathrm{d}t}.$$

Note that f depends on x and y [so partial derivatives $\partial f/\partial x$, $\partial f/\partial y$] whilst x and y depend on the single variable t [so ordinary derivatives $\mathrm{d}x/\mathrm{d}t$, $\mathrm{d}y/\mathrm{d}t$]. Thus f depends on t and has the derivative $\mathrm{d}f/\mathrm{d}t$ given above.

Example: If $f(x, y) = x^2 + y^2$, where $x = \sin t$, $y = t^3$, then

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \left(\frac{\partial f}{\partial x}\right)\frac{\mathrm{d}x}{\mathrm{d}t} + \left(\frac{\partial f}{\partial y}\right)\frac{\mathrm{d}y}{\mathrm{d}t} = 2x\cos t + 2y3t^2 = 2\sin t\cos t + 6t^5.$$

Of course in this simple example we can check the result by substituting for x and y before differentiation to give $f(t) = (\sin t)^2 + (t^3)^2$, so $\frac{\mathrm{d}f}{\mathrm{d}t} = 2\sin t\cos t + 6t^5$ as before.

The chain rule extends directly to functions of three or more variables.

1.4.1 Extended chain rule

For $f(x, y)$ suppose that x and y depend on **two** variables s and t. Then changing either s or t changes x and y, which in turn changes f. If we write

$$F(s, t) = f(x(s, t), y(s, t))$$

then the partial derivatives $\frac{\partial F}{\partial s}$ and $\frac{\partial F}{\partial t}$ are produced according to the *extended chain rule*

$$\frac{\partial F}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}, \qquad \frac{\partial F}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

Example: $f(x,y) = x^2y^3$, where $x = s - t^2$, $y = s + 2t$. Then

$$\frac{\partial f}{\partial x} = 2xy^3 \text{ and } \frac{\partial f}{\partial y} = 3x^2y^2$$

and

$$\frac{\partial F}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = 2xy^3 + 3x^2y^2 = (s-t^2)(s+2t)^2(5s+4t-3t^2)$$

$$\begin{aligned}\frac{\partial F}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = 2xy^3(-2t) + 3x^2y^2(2) \\ &= 2(s-t^2)(s+2t)^2(3s-2st-7t^2).\end{aligned}$$

Of course, we could just substitute in the definitions of x and y:

$$F(s,t) = (s-t^2)^2(s+2t)^3$$

which produces the same results. [Exercise: check this!]

Question:

For a function $f(x,y)$, if the independent variables x and y are changed to polar coordinates r and θ, so $x = r\cos\theta$, $y = r\sin\theta$, and $F(r,\theta) = f(x(r,\theta), y(r,\theta))$, show that

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

Solution:

We calculate the derivatives in terms of the polar coordinates:

$$\begin{aligned}\frac{\partial F}{\partial r} &= \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \\ \frac{\partial F}{\partial \theta} &= -\frac{\partial f}{\partial x}r\sin\theta + \frac{\partial f}{\partial y}r\cos\theta.\end{aligned}$$

If we wanted expressions for f_x and f_y we could combine these:

$$\begin{aligned}r\cos\theta\frac{\partial F}{\partial r} - \sin\theta\frac{\partial F}{\partial \theta} &= \frac{\partial f}{\partial x}r\cos^2\theta + \frac{\partial f}{\partial y}r\sin\theta\cos\theta + \frac{\partial f}{\partial x}r\sin^2\theta - \frac{\partial f}{\partial y}r\sin\theta\cos\theta = r\frac{\partial f}{\partial x} \\ \text{so} \quad \frac{\partial f}{\partial x} &= \cos\theta\frac{\partial F}{\partial r} - \frac{\sin\theta}{r}\frac{\partial F}{\partial \theta} \\ r\sin\theta\frac{\partial F}{\partial r} + \cos\theta\frac{\partial F}{\partial \theta} &= \frac{\partial f}{\partial x}r\sin\theta\cos\theta + \frac{\partial f}{\partial y}r\sin^2\theta - \frac{\partial f}{\partial x}r\sin\theta\cos\theta + \frac{\partial f}{\partial y}r\cos^2\theta = r\frac{\partial f}{\partial y} \\ \text{so} \quad \frac{\partial f}{\partial y} &= \sin\theta\frac{\partial F}{\partial r} + \frac{\cos\theta}{r}\frac{\partial F}{\partial \theta}.\end{aligned}$$

We calculate the second derivatives we need:

$$\begin{aligned}
\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right) &= \frac{\partial}{\partial x}\left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right)\cos\theta+\frac{\partial}{\partial y}\left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right)\sin\theta \\
\frac{\partial^2 F}{\partial\theta^2} &= \frac{\partial}{\partial x}\left(-y\frac{\partial f}{\partial x}+x\frac{\partial f}{\partial y}\right)(-r\sin\theta)+\frac{\partial}{\partial y}\left(-y\frac{\partial f}{\partial x}+x\frac{\partial f}{\partial y}\right)(r\cos\theta)
\end{aligned}$$

and simplify:

$$\begin{aligned}
\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right) &= \left(\frac{\partial f}{\partial x}+x\frac{\partial^2 f}{\partial x^2}+y\frac{\partial^2 f}{\partial x\partial y}\right)\cos\theta+\left(x\frac{\partial^2 f}{\partial x\partial y}+\frac{\partial f}{\partial y}+y\frac{\partial^2 f}{\partial y^2}\right)\sin\theta \\
\frac{\partial^2 F}{\partial\theta^2} &= -r\left(-y\frac{\partial^2 f}{\partial x^2}+\frac{\partial f}{\partial y}+x\frac{\partial^2 f}{\partial x\partial y}\right)\sin\theta+r\left(-\frac{\partial f}{\partial x}-y\frac{\partial^2 f}{\partial x\partial y}+x\frac{\partial^2 f}{\partial y^2}\right)\cos\theta
\end{aligned}$$

and add:

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2} &= \frac{1}{r}\left[\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)x\cos\theta+\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)y\sin\theta\right] \\
&= \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}
\end{aligned}$$

1.4.2 Matrix form of the extended chain rule

The form we had for the extended chain rule was

$$\begin{aligned}
\frac{\partial F}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial s}, \\
\frac{\partial F}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial t},
\end{aligned}$$

which we can write as a matrix-vector equation:

$$\begin{pmatrix} \partial F/\partial s \\ \partial F/\partial t \end{pmatrix} = \begin{pmatrix} \partial x/\partial s & \partial y/\partial s \\ \partial x/\partial t & \partial y/\partial t \end{pmatrix}\begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \end{pmatrix}$$

and the matrix in this equation is known as the **Jacobian matrix** of the transformation from s, t to x, y.

1.5 Change of Variables: Polar Coordinates

We are used to the three **Cartesian** or **rectangular** coordinates:

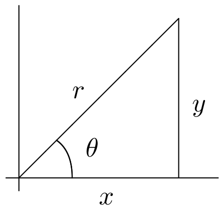
$$\begin{array}{ll} x & -\infty < x < \infty \\ y & -\infty < y < \infty \\ z & -\infty < z < \infty \end{array}$$

and we have also seen plane polar coordinates:

$$\begin{array}{ll} r & 0 \le r < \infty \\ \theta & 0 \le \theta < 2\pi \end{array}$$

which are related to x and y by

$$x = r\cos\theta \qquad y = r\sin\theta.$$



Example: Express in polar coordinates the portion of the unit disc that lies in the first quadrant.
Solution: The region may be expressed in polar coordinates as $0 \le r \le 1$ and $0 \le \theta \le \pi/2$.
Example: Express in polar coordinates the function

$$f(x,y) = x^2 + y^2 + 2yx.$$

Solution: We substitute $x = r\cos\theta$ and $y = r\sin\theta$ to have

$$f = r^2\cos^2\theta + r^2\sin^2\theta + 2r^2\sin\theta\cos\theta = r^2 + r^2\sin 2\theta.$$

1.5.1 Cylindrical Coordinates

These are really the three-dimensional equivalents of plane polar coordinates:

$$\begin{array}{ll} r & 0 \le r < \infty \\ \theta & 0 \le \theta < 2\pi \\ z & -\infty < z < \infty \end{array}$$

which are related to the rectangular coordinates by:

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z.$$

Example: Express in cylindrical coordinates the function

$$f(x,y,z) = x^2 + y^2 + z^2 - 2z\sqrt{(x^2+y^2)}$$

Solution: We substitute $x = r\cos\theta$ and $y = r\sin\theta$ to have

$$f = r^2\cos^2\theta + r^2\sin^2\theta + z^2 - 2z\sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = r^2 + z^2 - 2z\sqrt{r^2} = r^2 + z^2 - 2zr = (r-z)^2.$$

1.5.2 Spherical Coordinates

These are fully three-dimensional polar coordinates, and are used in lots of situations where there is a natural spherical symmetry (e.g. electron orbits).

$$\begin{array}{ll} \rho & 0 \le \rho < \infty \\ \theta & 0 \le \theta \le \pi \\ \phi & 0 \le \phi < 2\pi \end{array}$$

They are related to rectangular coordinates by

$$x = \rho \sin\theta \cos\phi \qquad y = \rho \sin\theta \sin\phi \qquad z = \rho \cos\theta$$

In the above equations, θ is the latitude or polar angle, and ϕ is the longitude.
Example: Express in spherical polars the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

Solution: We substitute the definitions $x = \rho \sin\theta \cos\phi$, $y = \rho \sin\theta \sin\phi$ and $z = \rho \cos\theta$ to get

$$\begin{aligned} f &= \rho^2 \sin^2\theta \cos^2\phi + \rho^2 \sin^2\theta \sin^2\phi + \rho^2 \cos^2\theta \\ &= \rho^2 \sin^2\theta [\cos^2\phi + \sin^2\phi] + \rho^2 \cos^2\theta = \rho^2 \sin^2\theta + \rho^2 \cos^2\theta = \rho^2. \end{aligned}$$

Example: Express in spherical polar coordinates the solid T that is bounded above by the cone $z^2 = x^2 + y^2$, below by the xy-plane, and on the sides by the hemisphere $z = (4 - x^2 - y^2)^{1/2}$.
Solution: The solid is defined by the following inequalities:

$$\begin{aligned} z^2 &\le x^2 + y^2 \\ z &\ge 0 \\ x^2 + y^2 + z^2 &\le 4 \end{aligned}$$

Substituting the definitions of x, y and z in terms of ρ, θ and ϕ gives:

$$\begin{aligned} \rho^2 \cos^2\theta &\le \rho^2 \sin^2\theta \cos^2\phi + \rho^2 \sin^2\theta \sin^2\phi \\ \rho \cos\theta &\ge 0 \\ \rho^2 \sin^2\theta \cos^2\phi + \rho^2 \sin^2\theta \sin^2\phi + \rho^2 \cos^2\theta &\le 4 \end{aligned}$$

so

$$\begin{aligned} \rho^2 \cos^2\theta &\le \rho^2 \sin^2\theta \\ \rho \cos\theta &\ge 0 \\ \rho^2 &\le 4 \end{aligned}$$

and we can use the fact that $\rho \ge 0$ and $\sin\theta \ge 0$ to deduce

$$\begin{aligned} \rho &\le 2 \\ \cos\theta &\ge 0 \\ 1 &\le \tan\theta. \end{aligned}$$

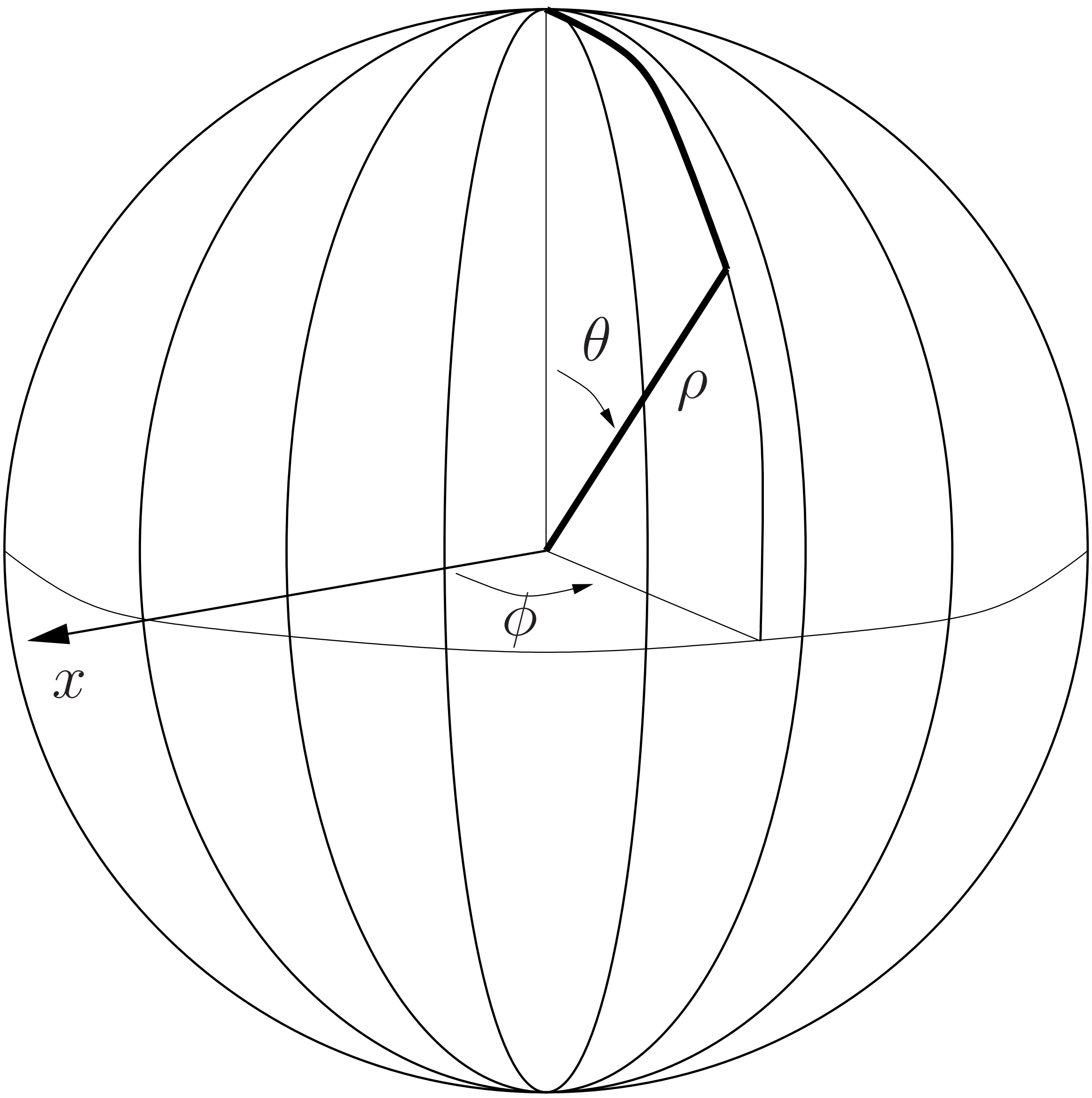


Figure 1: Spherical polar coordinates

Given that $0 \leq \theta \leq \pi$, this reduces to:

$$\begin{aligned} 0 \leq \ & \rho \ \leq 2 \\ \pi/4 \leq \ & \theta \ \leq \pi/2. \end{aligned}$$

In this case, where there is no information about ϕ contained in our limits, we use the whole permitted range:

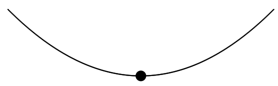
$$0 \leq \phi < 2\pi.$$

1.6 Critical Points

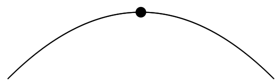
1.6.1 Maxima and Minima of a function of one variable

A **critical point** of an ordinary function $f(x)$ is a point at which $f'(x) = 0$, i.e. the graph is locally horizontal.

If $f''(x) > 0$ then the gradient is increasing and we have a local minimum:



If $f''(x) < 0$ then the gradient is decreasing and we have a local maximum:



If $f''(x) = 0$ then we may have any of three possibilities: a maximum, a minimum, or an inflexion point (e.g. $x = 0$ if $f(x) = x^3$). This is called a **degenerate** critical point and you won't need to classify any of these.

1.6.2 Critical Points of a Function of Two Variables

Definition: For a function of two variables, $f(x, y)$, a *critical point* is defined to be a point at which both of the first partial derivatives are zero:

$$\frac{\partial f}{\partial x} = 0, \qquad \frac{\partial f}{\partial y} = 0.$$

We can **classify** any critical point: this is the equivalent of, for an ordinary function, deciding whether it is a maximum, a minimum or an inflection point.

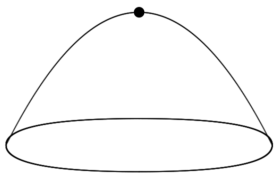
For a function of two variables, there are two key quantities we will need in order to classify our critical point:

- f_{xx}, the second partial derivative of f with respect to x, and
- $H = f_{xx}f_{yy} - f_{xy}^2$, the **Hessian**.

If the Hessian is zero, then our critical point is **degenerate**.

For a **non-degenerate** critical point, for which the Hessian is nonzero, there are three possible behaviours.

We may have a maximum:

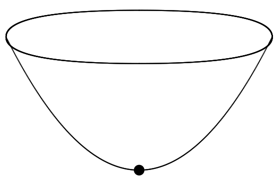


This happens if the Hessian is positive **and** f_{xx} (or f_{yy} if you prefer) is negative:

Sufficient conditions for a **maximum** at a critical point are that $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at that point.

The function decreases as you move away from the critical point in any direction.

We could have a minimum:

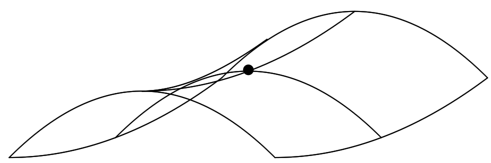


which happens when the Hessian is positive and so are f_{xx} and f_{yy}:

Sufficient conditions for a **minimum** at a critical point are that $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at that point.

The function increases as you move away from the critical point in any direction.

Finally, we may have a saddle point:



This happens if the Hessian is negative:

Sufficient condition for a **saddle point** is that $f_{xx}f_{yy} - f_{xy}^2 < 0$ at that point.

As you move away from the critical point, the function may increase or decrease depending on which direction you choose.

Example: Locate and classify the critical points of the function $f(x,y) = 12x^3 + y^3 + 12x^2y - 75y$.

Solution:

$$\begin{aligned} f_x &= 36x^2 + 24xy = 12x(3x + 2y), \\ f_y &= 3y^2 + 12x^2 - 75 = 3(4x^2 + y^2 - 25). \end{aligned}$$

Critical points are given by $f_x = 0$ and $f_y = 0$.
Now $f_x = 0 \implies x = 0$ or $3x + 2y = 0$.

(a) Suppose $x = 0$. Then $f_y = 3(y^2 - 25)$ so we need $y = \pm 5$.

(b) Otherwise, suppose $3x + 2y = 0$. Then $y = -3x/2$ and

$$f_y = 3(4x^2 + 9x^2/4 - 25) = (3/4)(16x^2 + 9x^2 - 100) = (3/4)(25x^2 - 100) = (75/4)(x^2 - 4)$$

so we need $x = \pm 2$.

We have found four critical points:

$$(0,5); \qquad (0,-5); \qquad (2,-3); \qquad (-2,3)$$

The 2nd order partial derivatives are

$$\begin{aligned} f_{xx} &= 72x + 24y = 24(3x + y), \\ f_{xy} &= 24x, \\ f_{yy} &= 6y. \end{aligned}$$

At $\boxed{(0,5)}$, $f_{xx} = 120 > 0,\ f_{xy} = 0,\ f_{yy} = 30,\ H = f_{xx}f_{yy} - f_{xy}^2 = 3600 > 0$, so this is a **minimum**.

At $\boxed{(0,-5)}$, $f_{xx} = -120 < 0,\ f_{xy} = 0,\ f_{yy} = -30,\ H = 3600 > 0$, so this is a **maximum**.

At $\boxed{(2,-3)}$, $f_{xx} = 72,\ f_{xy} = 48,\ f_{yy} = -18,\ H = -72 \times 18 - 48^2 < 0$, so this is a **saddle point**.

At $\boxed{(-2,3)}$, $f_{xx} = -72,\ f_{xy} = -48,\ f_{yy} = 18,\ H = -72 \times 18 - 48^2 < 0$, so this is a **saddle point**.

2 Grad, Div, Curl and all that

2.1 Gradient vector in two dimensions

For a function of two variables $f(x, y)$, we have seen that the function can be used to represent the surface

$$z = f(x, y)$$

and the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ measure the slope of the surface along the x and y directions, respectively.

We now ask how we can calculate the slope of f in **any** direction in space. The answer lies in the vector

$$\underline{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j}$$

called the *gradient* of f.

Example

If $f(x, y) = x^2y^2 + x^3 + y$, find $\underline{\nabla} f$ at the point $x = 2$, $y = 5$.

Solution

We first work out the first partial derivatives:

$$\begin{aligned} f_x &= 2xy^2 + 3x^2 \\ f_y &= 2x^2y + 1 \end{aligned}$$

to give

$$\underline{\nabla} f = \left(2xy^2 + 3x^2, 2x^2y + 1\right)$$

and then substitute in the values:

$$\underline{\nabla} f = (112, 41)\,.$$

Example

If we know

$$\underline{\nabla} f = \left(3x^2y^2 + x^3, 2x^3y + \cos y\right)$$

find the most general possible form of $f(x, y)$.

Solution

We start by using the definition of $\underline{\nabla} f$ to separate this into two equations:

$$\begin{aligned} \frac{\partial f}{\partial x} &= 3x^2y^2 + x^3 \\ \frac{\partial f}{\partial y} &= 2x^3y + \cos y \end{aligned}$$

Now we integrate one of them, remembering that for a partial derivative, all the other variables act like constants, so when we integrate a partial derivative our "constant of integration" will depend on all the other variables.

$$\frac{\partial f}{\partial x} = 3x^2y^2 + x^3 \implies f(x, y) = x^3y^2 + \frac{1}{4}x^2 + g(y).$$

Now we can differentiate this with respect to y:

$$\frac{\partial f}{\partial y} = 2x^3 y + \frac{\mathrm{d}g}{\mathrm{d}y}.$$

Notice that since g only depends on y this is now an ordinary derivative. We already know that

$$\frac{\partial f}{\partial y} = 2x^3 y + \cos y$$

so to make these consistent we need

$$2x^3 y + \cos y = 2x^3 y + \frac{\mathrm{d}g}{\mathrm{d}y} \implies \frac{\mathrm{d}g}{\mathrm{d}y} = \cos y$$

and this is now an ordinary integration, giving as its result

$$g(y) = \sin y + c.$$

The final answer is then

$$f(x, y) = x^3 y^2 + \frac{1}{4} x^2 + \sin y + c.$$

2.2 Directional Derivative

First of all notice that

$$\underline{i} \cdot \underline{\nabla} f = \frac{\partial f}{\partial x}$$

dotting $\underline{\nabla} f$ with the unit vector in the x direction gives the slope in the x direction.
In the same way,

$$\underline{j} \cdot \underline{\nabla} f = \frac{\partial f}{\partial y}$$

dotting $\underline{\nabla} f$ with the unit vector in the y direction gives the slope in the y direction.
We can do the same thing with any other direction: the *directional derivative*

$$\underline{u} \cdot \underline{\nabla} f$$

gives the slope of the surface measured in the direction of the unit vector $\underline{u}$.
Example: If $f(x, y) = x^2 + xy$, find $\underline{\nabla} f$. What is the slope of the surface $z = f(x, y)$ along the direction $\underline{i} + 2\underline{j}$ at the point $(1, 1)$?
Solution:

$$\underline{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (2x + y, x).$$

Now at the point $(1, 1)$, we have $\underline{\nabla} f = (3, 1)$. To find the slope of f along a vector $\underline{v}$, we need to calculate the dot product of $\underline{\nabla} f$ with the unit vector of $\underline{v}$. Here, $\underline{v} = (1, 2)$ which has modulus $\sqrt{(1 + 4)} = \sqrt{5}$ so the unit vector is $\underline{u} = (1/\sqrt{5}, 2/\sqrt{5})$.

$$\underline{u} \cdot \underline{\nabla} f = (1/\sqrt{5}, 2/\sqrt{5}) \cdot (3, 1) = 3/\sqrt{5} + 2/\sqrt{5} = \sqrt{5}.$$

2.2.1 Two properties of the gradient in two dimensions

We are looking at a function $f(x, y)$ which represents the surface $z = f(x, y)$.

Property 1. At any point, $\underline{\nabla} f$ points in the direction in which f is increasing most rapidly: i.e. $\underline{\nabla} f$ points uphill. Its magnitude $|\underline{\nabla} f|$ gives the slope in this steepest direction.

Property 2. At any point, $\underline{\nabla} f$ is perpendicular to the contour line $f =$ const. through that point.

2.3 Gradient vector in three dimensions

Now let us look at a function of three variables, $f(x, y, z)$. We can still calculate the gradient vector:

$$\underline{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j} + \frac{\partial f}{\partial z}\underline{k}.$$

We can still represent a surface using our function: the equation

$$f(x, y, z) = A$$

describes a surface in three-dimensional space for each value of A.

Examples

$$x + y + z = 1$$

represents a plane: it can also be written as

$$z = 1 - x - y.$$

$$x^2 + y^2 - z = 0$$

can be written as

$$z = x^2 + y^2$$

which is a surface we have already seen: the paraboloid bowl.

Finally, if we have

$$f(x, y, z) = x^2 + y^2 + z^2$$

then the equation

$$f(x, y, z) = 4$$

represents the sphere centred on the origin of radius 2. This is easier to see if we use spherical polar coordinates:

$$f = \rho^2 \qquad \rho = 2.$$

Remember in 2D we had two properties: $\underline{\nabla} f$ points uphill, and $\underline{\nabla} f$ is perpendicular to contour lines. There are equivalent properties in 3D:

- $\underline{\nabla} f$ points in the direction in which f increases fastest, and its magnitude gives the rate of change of f in that direction.

- $\underline{\nabla} f$ is perpendicular to the surface $f =$ constant.

Let's look at our example function $f(x, y, z) = x^2 + y^2 + z^2$.

$$\underline{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2x, 2y, 2z) = 2(x, y, z).$$

This is double the position vector, so it points in the radial direction. Now remember that the surface $f(x, y, z) =$ constant represented a sphere, and we know intuitively that the perpendicular to the surface of a sphere points outwards along the radius. On the earth, the vector perpendicular to the surface is vertical, which is along the same line as the centre of the earth. So this agrees with the property that $\underline{\nabla} f$ is perpendicular to the surface $f =$ constant.

Example: Find a vector perpendicular to the surface $z = x^2 + y^2$ at the point $(1, 2, 5)$.

Solution:

The vector $\underline{\nabla} f$ is perpendicular to the surface $f =$ const. so we need to write our surface in the form $f =$ const. We use

$$f(x, y, z) = x^2 + y^2 - z$$

and our surface is $f(x, y, z) = 0$.

$$\underline{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2x, 2y, -1)$$

At the point $(1, 2, 5)$ we have $x = 1$, $y = 2$, $z = 5$ so the gradient is

$$\underline{\nabla} f = (2, 4, -1)$$

and this vector is perpendicular to the surface.

2.4 Vector fields

What we have from the gradient is a **vector function** or **vector field**: for each point (x, y, z) it gives a vector.

A vector field does not have to be a gradient: in the same way that we can have an ordinary function of either one variable:

$$f(x)$$

or of more:

$$f(x, y, z)$$

we can form a general vector function of three variables

$$\underline{v}(\underline{x}) = \underline{v}(x, y, z) = (v_1(x, y, z), v_2(x, y, z), v_3(x, y, z))$$

For every point in 3D space, this vector field assigns a vector.

Example:

$$\underline{v}(\underline{x}) = (y, x, x^2 + y^2 + z^2)$$

Let us look at a few values.

$$\underline{v}(0,0,0) = (0,0,0) \qquad \underline{v}(1,0,0) = (0,1,1)$$
$$\underline{v}(0,1,0) = (1,0,1) \qquad \underline{v}(0,0,1) = (0,0,1)$$
$$\underline{v}(1,1,1) = (1,1,3) \qquad \underline{v}(1,2,1) = (2,1,6)$$

As you can see, the first component of the *input* or *argument* vector $\underline{x}$ is x, the second is y and the third is z. We substitute in the three values for each component to get the three components of the output vector $\underline{v}$.

Examples

Some physical uses of vector functions are:

- Magnetic field: this has a magnitude and a direction so it is a vector, and it can be different at every point in space
- Fluid velocity: think about a turbulent river flow – there are many different velocities at different points in space
- Temperature gradient: the heat flow through one point of a conducting object is proportional to the gradient of temperature at that point.
- Normal to a surface: if we look at a smooth 2D surface in 3D space, at any point on the surface we can find a vector which is perpendicular to the surface. This gives us a (different) vector for every point on the surface: in other words, a vector which is a function of position, or a vector field again. This is what we did in the last example of the section on $\underline{\nabla}$, and is called the **normal field** to the surface.
- Tangent to a surface: at each point of a surface, there is more than one vector parallel to the surface (in fact there is a plane of them); but we can write down a vector function which is at every point tangent to (parallel to) the surface and this is called a **tangent field**.

2.5 Grad as a separate entity

Suppose that $f(x,y,z)$ is a scalar function. Then we can think of the $\underline{\nabla}$ part of $\underline{\nabla} f$ separately, as an **operator** (more on these later). It operates on f as follows:

$$\underline{\nabla} f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).$$

This is the *gradient* of f, which we have just discussed. It takes a scalar function and gives us a vector field. The operator $\underline{\nabla}$ is given by

$$\underline{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

and it is not really a vector in its own right; it only exists to operate on (or apply to) something else.

Question How does $\underline{\nabla}$ operate on vector functions or vector fields?

Answer Using either of the vector products we already know – dot or cross.

2.6 Divergence

If $\underline{q}(x,y,z) = (q_1(x,y,z), q_2(x,y,z), q_3(x,y,z))$ is a vector function, then by definition

$$div(\underline{q}) = \underline{\nabla} \cdot \underline{q} = \frac{\partial q_1}{\partial x} + \frac{\partial q_2}{\partial y} + \frac{\partial q_3}{\partial z}. \tag{1}$$

This "product", $\underline{\nabla} \cdot \underline{q}$, defined in imitation of the ordinary dot product, is a scalar called the *divergence* of $\underline{\text{q}}$.

Example: Calculate $\underline{\nabla} \cdot \underline{q}$ for the vector function $\underline{q}(x,y,z) = (x-y, x+y, z)$.

Solution:

$$\underline{\nabla} \cdot \underline{q} = \frac{\partial (x-y)}{\partial x} + \frac{\partial (x+y)}{\partial y} + \frac{\partial z}{\partial z} = 1+1+1 = 3.$$

2.7 Curl

For the vector $\underline{q} = (q_1(x,y,z), q_2(x,y,z), q_3(x,y,z))$, we also have

$$\begin{aligned} \underline{curl}(\underline{q}) = \underline{\nabla} \times \underline{q} &= det \begin{bmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ q_1 & q_2 & q_3 \end{bmatrix} \\ &= \left(\frac{\partial q_3}{\partial y} - \frac{\partial q_2}{\partial z}, \frac{\partial q_1}{\partial z} - \frac{\partial q_3}{\partial x}, \frac{\partial q_2}{\partial x} - \frac{\partial q_1}{\partial y} \right). \end{aligned} \tag{2}$$

This second "product", $\underline{\nabla} \times \underline{q}$, defined in imitation of the ordinary cross product, is a <u>vector</u> called the *curl* of $\underline{q}$.

Example: Calculate $\underline{\nabla} \times \underline{q}$ for (from above) $\underline{q}(x,y,z) = (x-y, x+y, z)$.

Solution:

$$\underline{\nabla} \times \underline{q} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x-y & x+y & z \end{vmatrix} = \underline{i}(0) + \underline{j}(0) + \underline{k}(1+1) = 2\underline{k}.$$

2.8 Laplacian

Now we have two operators, based on $\underline{\nabla}$, that we can apply to any vector field: div and curl. Since $\underline{\nabla} f$ is a vector field for any scalar function $f(x,y,z)$, we can apply either of them to $\underline{\nabla} f$. What do we get?

Curl of gradient

This one is not very interesting:

$$curl(\underline{\nabla} f) = \underline{\nabla} \times \underline{\nabla} f = \underline{0}.$$

Div of gradient

Because it's based on the dot product, this gives a scalar:

$$div(\underline{\nabla} f) = \underline{\nabla} \cdot \underline{\nabla} f = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = f_{xx} + f_{yy} + f_{zz}.$$

This is called the **Laplacian** and is used in lots of applications.

$$\nabla^2 f = f_{xx} + f_{yy} + f_{zz}.$$

Example: Calculate the Laplacian for $f(x, y, z) = x^2 + y^2 + z^2$.
Solution:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial x}2x + \frac{\partial}{\partial y}2y + \frac{\partial}{\partial z}2z = 2 + 2 + 2 = 6.$$

The forms for the Laplacian in different coordinate systems are more complex (we found one of these as an example for polar coordinates). They are given by:

$$\begin{aligned}
\nabla^2 f &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} \text{ in plane polar coordinates} \\
\nabla^2 f &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \text{ in cylindrical polar coordinates} \\
\nabla^2 f &= \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{\rho^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} \text{ in spherical polar coordinates}
\end{aligned}$$

Example: Looking again at the example we used for the Laplacian:

$$f(x, y, z) = x^2 + y^2 + z^2 \qquad \nabla^2 f = 6.$$

We will recalculate this in the different coordinate systems.
In cylindrical polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$ so

$$f(r, \theta, z) = r^2\cos^2\theta + r^2\sin^2\theta + z^2 = r^2 + z^2.$$

Then using the formula for ∇^2 in cylindrical polars:

$$\begin{aligned}
\nabla^2 f &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}(r\,2r) + 0 + \frac{\partial}{\partial z}(2z) \\
&= \frac{1}{r}\frac{\partial}{\partial r}(2r^2) + \frac{\partial}{\partial z}(2z) = \frac{1}{r}(4r) + 2 = 4 + 2 = 6.
\end{aligned}$$

In spherical polar coordinates, $x = \rho\sin\theta\cos\phi$, $y = \rho\sin\theta\sin\phi$ and $z = \rho\cos\theta$ so

$$f(\rho, \theta, \phi) = \rho^2.$$

Then using the formula for ∇^2 in spherical polars:

$$\begin{aligned}
\nabla^2 f &= \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{\rho^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} \\
&= \frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\rho^2\,2\rho) + 0 + 0 = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}(2\rho^3) = \frac{1}{\rho^2}6\rho^2 = 6.
\end{aligned}$$

These three different calculations all produce the same result because ∇^2 is a derivative with a real physical meaning, and does not depend on what coordinate system is being used to look at the system.

2.9 Real Partial Differential Equations

Remember the examples we had of partial differential equations. Here we will run through most of them again in vector form, using the functions div, grad, curl and the Laplacian:

Quantum mechanics We can re-write **Schrödinger's equation** as

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(x,y,z)\psi.$$

Electromagnetism In the scalar format there were eight **Maxwell's equations**: four of them are covered by these two vector equations:

$$\underline{\nabla}\cdot\underline{B} = 0$$

$$\frac{\partial\underline{B}}{\partial t} + \underline{\nabla}\times\underline{E} = \underline{0}.$$

Fluid flow The **Navier-Stokes equations** in vector form are:

$$\rho\left(\frac{\partial\underline{u}}{\partial t} + (\underline{u}\cdot\underline{\nabla})\underline{u}\right) = -\underline{\nabla}p + \mu\nabla^2\underline{u}.$$

Heat transfer The heat equation becomes

$$\frac{\partial T}{\partial t} = \kappa\nabla^2 T.$$

3 Operators and the Commutator

3.1 Operators: Introduction and examples

An **operator** is an instruction or action: it operates on something.
Example: for a variable x, the function $f(x) = ax$ can be thought of as an operator on x:

$$f \quad : \quad x \to ax \qquad \text{"multiply by } a\text{"}$$

and so can any other function, e.g. $g(x) = x^2$:

$$g \quad : \quad x \to x^2 \qquad \text{"square"}$$

Equally, differentiation can be thought of as an operator acting on a function of x:

$$D \quad : \quad f(x) \to \frac{\mathrm{d}f}{\mathrm{d}x} \qquad \text{"differentiate with respect to } x\text{"}$$

For a vector $\underline{v}$, a matrix $\underline{\underline{A}}$ is an operator on $\underline{v}$:

$$\underline{\underline{A}} \quad : \quad \underline{v} \to \underline{\underline{A}}\,\underline{v} \qquad \text{"multiply by } \underline{\mathrm{A}}\text{"}$$

Another operator that can operate on a function is multiplication by a constant:

$$a \quad : \quad f(x) \to af(x) \qquad \text{"multiply by } a\text{"}$$

3.2 Linear operators

$O(f)$ is a linear operator if:

- $O(f + g) = O(f) + O(g)$ and
- $O(\lambda f) = \lambda O(f)$

We can show whether an operator is linear or not by checking these two properties.

Multiply by a

$$\begin{aligned} O \quad &: \quad f(x) \to af(x) \\ O(f+g) &= a(f+g) = af(x) + ag(x) = O(f) + O(g) \\ O(\lambda f) &= a\lambda f(x) = \lambda a f(x) = \lambda O(f). \end{aligned}$$

We have checked both properties so it is linear.

Multiply by x

$$\begin{aligned} O \quad &: \quad f(x) \to xf(x) \\ O(f+g) &= x(f+g) = xf(x) + xg(x) = O(f) + O(g) \\ O(\lambda f) &= x\lambda f(x) = \lambda x f(x) = \lambda O(f). \end{aligned}$$

We have checked both properties so it is linear.

Differentiate with respect to x

$$\begin{aligned} D &: f(x) \to \frac{\mathrm{d}f}{\mathrm{d}x} \\ D(f+g) &= \frac{\mathrm{d}}{\mathrm{d}x}(f(x)+g(x)) = \frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\mathrm{d}g}{\mathrm{d}x} = D(f) + D(g) \\ D(\lambda f) &= \frac{\mathrm{d}}{\mathrm{d}x}(\lambda f(x)) = \lambda \frac{\mathrm{d}f}{\mathrm{d}x} = \lambda D(f). \end{aligned}$$

We have checked both properties so it is linear.

Squaring function

$$\begin{aligned} f &: x \to x^2 \\ f(x+y) &= (x+y)^2 = x^2 + 2xy + y^2 = f(x) + f(y) + 2xy \end{aligned}$$

so the squaring operator is **not** linear.

Multiply by x^2 function

$$\begin{aligned} O &: f(x) \to x^2 f(x) \\ O(f+g) &= x^2(f+g) = x^2 f + x^2 g = O(f) + O(G) \\ O(\lambda f) &= x^2(\lambda f) = \lambda x^2 f = \lambda O(f) \end{aligned}$$

so this operator is linear.

3.3 Composing operators

If we have two different operators and we want to apply them to a function in sequence, we use the notation

$$O_1 \circ O_2 \quad : \quad f \to O_1(O_2(f))$$

so we apply O_2 first, then apply O_1 to the result.

Example

$$\begin{aligned} O_A &: f(x) \to x f(x) \\ O_B &: g(x) \to g^2(x) \end{aligned}$$

We will compute $O_A \circ O_B$:

$$\begin{aligned} O_A \circ O_B &: g(x) \xrightarrow{B} g^2(x) \xrightarrow{A} x g^2(x) \\ O_A \circ O_B &: g(x) \to x g^2(x) \end{aligned}$$

Suppose $g(x) = e^x$: then

$$\begin{aligned} O_A(g) &= xe^x \\ O_B(g) &= e^{2x} \\ O_A \circ O_B(g) &= xe^{2x} \end{aligned}$$

Example

$$\begin{aligned} O_A &: f(x) \to \mathrm{d}f/\mathrm{d}x + f(x) \\ O_B &: f(x) \to \mathrm{d}f/\mathrm{d}x + 2f(x) \end{aligned}$$

We can also write these as

$$O_A = D + 1 \qquad O_B = D + 2.$$

We will compute $O_A \circ O_B$:

$$\begin{aligned} O_A \circ O_B &: f(x) \xrightarrow{B} \mathrm{d}f/\mathrm{d}x + 2f(x) \xrightarrow{A} (\mathrm{d}/\mathrm{d}x + 1)\,(\mathrm{d}f/\mathrm{d}x + 2f(x)) \\ &= \mathrm{d}^2 f/\mathrm{d}x^2 + 2\mathrm{d}f/\mathrm{d}x + \mathrm{d}f/\mathrm{d}x + 2f(x) \\ &= \mathrm{d}^2 f/\mathrm{d}x^2 + 3\mathrm{d}f/\mathrm{d}x + 2f(x) \\ O_A \circ O_B &= D^2 + 3D + 2 \end{aligned}$$

and similarly, (left as an exercise to the reader):

$$O_B \circ O_A \quad : \quad f(x) \to (D^2 + 3D + 2)f(x)$$

i.e. the same as $O_A \circ O_B$, but this will not always work.
We can apply derivatives repeatedly, but we can also mix our operators.

$$\begin{aligned} D &: f(x) \to \frac{\mathrm{d}f}{\mathrm{d}x} \\ D^2 = D \circ D &: f(x) \to D(\frac{\mathrm{d}f}{\mathrm{d}x}) = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \\ (D + a) &: f(x) \to \frac{\mathrm{d}f}{\mathrm{d}x} + af(x) \\ (D + a) \circ (D + b) &: f(x) \to (D + a)\left(\frac{\mathrm{d}f}{\mathrm{d}x} + bf(x)\right) \\ &= \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \frac{\mathrm{d}}{\mathrm{d}x}(bf(x)) + a\frac{\mathrm{d}f}{\mathrm{d}x} + abf(x) \\ &= \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + (a + b)\frac{\mathrm{d}f}{\mathrm{d}x} + abf(x) \\ &= (D^2 + (a + b)D + ab)f \end{aligned}$$

It doesn't have to be all constants and derivatives:

$$\begin{aligned} D \circ x &: f(x) \to D(xf(x)) = \frac{\mathrm{d}}{\mathrm{d}x}(xf(x)) = f(x) + x\frac{\mathrm{d}f}{\mathrm{d}x} \\ D \circ x(x^2) &= x^2 + x(2x) = x^2 + 2x^2 = 3x^2 = D(x^3) \end{aligned}$$

We must be careful about the order of our operators though:

$$x \circ D \quad : \quad f(x) \to x(Df(x)) = x\frac{\mathrm{d}f}{\mathrm{d}x} \neq D \circ x(f)$$

A couple more linearity checks:

Second derivative D^2

$$
\begin{aligned}
D^2 &: \quad f(x) \to \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \\
D^2(f+g) &= \frac{\mathrm{d}^2}{\mathrm{d}x^2}(f+g) = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \frac{\mathrm{d}^2 g}{\mathrm{d}x^2} = D^2 f + D^2 g \\
D^2(\lambda f) &= \frac{\mathrm{d}^2}{\mathrm{d}x^2}(\lambda f) = \lambda D^2 f
\end{aligned}
$$

Just as D was linear, so is D^2.

The operator xD

$$
\begin{aligned}
xD &: \quad f(x) \to x\frac{\mathrm{d}f}{\mathrm{d}x} \\
xD(f+g) &= x\frac{\mathrm{d}}{\mathrm{d}x}(f+g) = x\frac{\mathrm{d}f}{\mathrm{d}x} + x\frac{\mathrm{d}g}{\mathrm{d}x} = xDf + xDg \\
xD(\lambda f) &= x\frac{\mathrm{d}}{\mathrm{d}x}(\lambda f) = x\lambda\frac{\mathrm{d}f}{\mathrm{d}x} = \lambda xDf
\end{aligned}
$$

so the operator xD is also linear.

In general, if O_A is linear and O_B is linear, then $O_A \circ O_B$ and $O_B \circ O_A$ are also linear.

3.4 Partial derivatives

Partial derivatives can also be treated as operators:

$$
\begin{aligned}
D_x &: \quad f(x,y) \to \frac{\partial f}{\partial x} \\
D_y &: \quad f(x,y) \to \frac{\partial f}{\partial y}
\end{aligned}
$$

Just like the ordinary derivative D, these are linear operators. Equally, $\underline{\nabla}$ is an operator, as are div, curl and the Laplacian:

$$
\begin{aligned}
\underline{\nabla} &: \quad f(x,y,z) \to \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\
div &: \quad \underline{q}(x,y,z) \to \frac{\partial q_1}{\partial x} + \frac{\partial q_2}{\partial y} + \frac{\partial q_3}{\partial z} \\
curl &: \quad \underline{q}(x,y,z) \to \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ q_1 & q_2 & q_3 \end{vmatrix} \\
\nabla^2 &: \quad f(x,y,z) \to \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}
$$

All four of these are also linear.

3.5 Order of operators is important

Remember if we have two different operators O_A and O_B, we can apply them in turn:

$$O_A \circ O_B \quad : \quad f \to O_A(O_B(f))$$

Let us look at an example:

$$\begin{aligned} O_A &: \quad f(x) \to \frac{\mathrm{d}f}{\mathrm{d}x} \\ O_B &: \quad f(x) \to x\frac{\mathrm{d}f}{\mathrm{d}x} \end{aligned}$$

$$\begin{aligned} O_A \circ O_B &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right) \\ O_B \circ O_A &= x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) \end{aligned}$$

The order can be important:

$$\begin{aligned} O_A \circ O_B &= \frac{\mathrm{d}}{\mathrm{d}x} + x\frac{\mathrm{d}^2}{\mathrm{d}x^2} \\ O_B \circ O_A &= x\frac{\mathrm{d}^2}{\mathrm{d}x^2} \neq O_A \circ O_B \end{aligned}$$

For example, try applying them to $f(x) = e^x$.

$$\begin{aligned} O_A \circ O_B(e^x) &= \left(\frac{\mathrm{d}}{\mathrm{d}x} + x\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)e^x = e^x + xe^x = (1+x)e^x \\ O_B \circ O_A(e^x) &= x\frac{\mathrm{d}^2}{\mathrm{d}x^2}e^x = xe^x \end{aligned}$$

Let us look at another example two operators: this time two matrices acting as linear operators on a vector:

$$\begin{aligned} O_A &: \quad \underline{v} \to \underline{\underline{A}}\,\underline{v} \\ O_B &: \quad \underline{v} \to \underline{\underline{B}}\,\underline{v} \end{aligned}$$

with

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \underline{\underline{B}} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

The composed operators are:

$$\begin{aligned} O_A \circ O_B &: \quad \underline{v} \to \underline{\underline{A}}\,\underline{\underline{B}}\,\underline{v} \\ O_B \circ O_A &: \quad \underline{v} \to \underline{\underline{B}}\,\underline{\underline{A}}\,\underline{v} \end{aligned}$$

and the matrices involved there:

$$\underline{\underline{A}}\,\underline{\underline{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\underline{\underline{B}}\,\underline{\underline{A}} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

Since the order of matrix multiplication is important, this is another case where the order of the operators is important.

3.6 The commutator

Given two operators O_A and O_B, the **commutator** of the two is

$$[O_A, O_B] = O_A \circ O_B - O_B \circ O_A.$$

It can be nonzero precisely because the order of operators is important.

Let us work out the commutators for the examples from the previous section. We start with the matrix example:

$$\begin{aligned} O_A &: \underline{v} \to \underline{\underline{A}}\,\underline{v} \\ O_B &: \underline{v} \to \underline{\underline{B}}\,\underline{v} \end{aligned}$$

with

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \underline{\underline{B}} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

and we had worked out the composed operators:

$$\begin{aligned} O_A \circ O_B &: \underline{v} \to \underline{\underline{A}}\,\underline{\underline{B}}\,\underline{v} \\ O_B \circ O_A &: \underline{v} \to \underline{\underline{B}}\,\underline{\underline{A}}\,\underline{v} \end{aligned}$$

with the matrix products:

$$\underline{\underline{A}}\,\underline{\underline{B}} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad \underline{\underline{B}}\,\underline{\underline{A}} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

The commutator is thus:

$$[O_A, O_B] = O_A \circ O_B - O_B \circ O_A$$

$$[O_A, O_B] \quad : \quad \underline{v} \to (\underline{\underline{A}}\,\underline{\underline{B}} - \underline{\underline{B}}\,\underline{\underline{A}})\underline{v}$$

and we can write this as

$$[O_A, O_B] \quad : \quad \underline{v} \to \underline{\underline{C}}\,\underline{v}$$

with

$$\underline{\underline{C}} = \underline{\underline{A}}\,\underline{\underline{B}} - \underline{\underline{B}}\,\underline{\underline{A}} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Note that the commutator of two operators is itself an operator.

Next we look at a commutator involving a derivative:

$$\begin{aligned} D &: f(x) \to \frac{\mathrm{d}f}{\mathrm{d}x} \\ xD &: f(x) \to x\frac{\mathrm{d}f}{\mathrm{d}x} \end{aligned}$$

The commutator is

$$[D, xD] = D \circ xD - xD \circ D = D + xD^2 - xD^2 = D.$$

Let us work out just one more example for practice:

$$\begin{aligned}
[x^2, D^2] &= x^2 \circ D^2 - D^2 \circ x^2 \\
&= x^2 D^2 - D \circ D \circ x^2 \\
&= x^2 D^2 - D \circ (2x + x^2 D) \\
&= x^2 D^2 - (2 + 2xD + 2xD + x^2 D^2) \\
&= -2 - 4xD
\end{aligned}$$

Remember, x^2 as an operator means "multiply by x^2".

3.7 Commutators and Partial Derivatives

We can involve both partial derivatives and multiple variables in a commutator: e.g.

$$\begin{aligned}
\left[\frac{\partial}{\partial x}, xy\right] &: \quad f \to \frac{\partial}{\partial x}(xyf) - xy\frac{\partial f}{\partial x} \\
&= \quad yf + xy\frac{\partial}{\partial x}(f) - xy\frac{\partial f}{\partial x} = yf
\end{aligned}$$

so

$$\left[\frac{\partial}{\partial x}, xy\right] = y.$$

Similarly,

$$\begin{aligned}
\left[x\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}\right] &: \quad f \to x\frac{\partial}{\partial y}\left(y\frac{\partial f}{\partial x}\right) - y\frac{\partial}{\partial x}\left(x\frac{\partial f}{\partial y}\right) \\
&= \quad x\frac{\partial f}{\partial x} + xy\frac{\partial^2 f}{\partial x \partial y} - y\frac{\partial f}{\partial y} - yx\frac{\partial^2 f}{\partial y \partial x} \\
&= \quad x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}
\end{aligned}$$

so

$$\left[x\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}\right] = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

The next example comes from classical mechanics. The angular momentum of a particle with unit mass travelling with velocity $\underline{\nabla}\phi$ is $\underline{r} \times \underline{\nabla}\phi$: you don't need to worry about this but let's work out the components.

$$\underline{r} \times \underline{\nabla}\phi = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x & y & z \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \end{vmatrix} = \underline{i}\left(y\frac{\partial \phi}{\partial z} - z\frac{\partial \phi}{\partial y}\right) + \underline{j}\left(z\frac{\partial \phi}{\partial x} - x\frac{\partial \phi}{\partial z}\right) + \underline{k}\left(x\frac{\partial \phi}{\partial y} - y\frac{\partial \phi}{\partial x}\right)$$

A handy shorthand is to number the variables from 1 to 3:

$$x_1 = x \qquad x_2 = y \qquad x_3 = z.$$

Then we can define the **angular momentum operator**

$$\begin{aligned} L_{ab} &: \phi \to x_a \frac{\partial \phi}{\partial x_b} - x_b \frac{\partial \phi}{\partial x_a} \\ L_{ab} &= x_a \frac{\partial}{\partial x_b} - x_b \frac{\partial}{\partial x_a} \end{aligned}$$

This means that our angular momentum vector is

$$\underline{r} \times \underline{\nabla}\phi = \underline{i}L_{23}(\phi) + \underline{j}L_{31}(\phi) + \underline{k}L_{12}(\phi).$$

and we could even treat the whole thing as an operator on ϕ:

$$\underline{r} \times \underline{\nabla} = \underline{i}L_{23} + \underline{j}L_{31} + \underline{k}L_{12}.$$

Returning to the scalar operators L_{ab}, we can work out the commutator of two of them:

$$\begin{aligned} [L_{12}, L_{23}] &= (xD_y - yD_x)(yD_z - zD_y) - (yD_z - zD_y)(xD_y - yD_x) \\ &= xD_y(yD_z - zD_y) - yD_x(yD_z - zD_y) - yD_z(xD_y - yD_x) + zD_y(xD_y - yD_x) \\ &= x(D_z + yD_yD_z - zD_y^2) - y(yD_xD_z - zD_xD_y) \\ &\qquad - y(xD_zD_y - yD_zD_x) + z(xD_y^2 - D_x - yD_yD_x) \\ &= xD_z - zD_x = L_{13}. \end{aligned}$$

3.8 Extended Chain Rule Revisited

Remember the extended chain rule: if $x = x(s,t)$ and $y = y(s,t)$, and we have a function $f(x,y)$ so that

$$F(s,t) = f(x(s,t), y(s,t))$$

then

$$\begin{aligned} \frac{\partial F}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \\ \frac{\partial F}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}. \end{aligned}$$

We can write this in operator form as

$$\begin{aligned} D_s &= \frac{\partial x}{\partial s}D_x + \frac{\partial y}{\partial s}D_y \\ D_t &= \frac{\partial x}{\partial t}D_x + \frac{\partial y}{\partial t}D_y. \end{aligned}$$

Let us look again at the example of calculating the form of the operator ∇^2 in plane polar coordinates. Let us use s for r and t for θ:

$$x = s\cos t \qquad y = s\sin t$$

then the extended chain rule gives us

$$\begin{aligned} D_s &= \cos t D_x + \sin t D_y \\ D_t &= -s \sin t D_x + s \cos t D_y. \end{aligned}$$

We will start from the standard form:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

which gives us the linear operator

$$L \quad : \quad f \to \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial f}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 f}{\partial t^2}$$

or

$$L = \frac{1}{s}D_s(sD_s) + \frac{1}{s^2}D_t^2$$

Now we can look at the individual terms:

$$\begin{aligned} \frac{1}{s}D_s(sD_s) &= \frac{1}{s}D_s(s[\cos t D_x + \sin t D_y]) \\ &= \frac{1}{s}(\cos t D_x + \sin t D_y)(s \cos t D_x + s \sin t D_y) \\ &= \frac{1}{s}(\cos t D_x + \sin t D_y)(xD_x + yD_y) \ldots \end{aligned}$$

The rest of the calculation is left as an exercise.

4 Ordinary Differential Equations

4.1 Introduction

A **differential equation** is an equation relating an *independent* variable, e.g. t, a *dependent* variable, y, and one or more derivatives of y with respect to t:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3x \qquad y^2\frac{\mathrm{d}y}{\mathrm{d}t} = e^t \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3x^2y^2\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

In this section we will look at some specific types of differential equation and how to solve them.

4.2 Classifying equations

We can classify our differential equation by four properties:

- Is it an **ordinary** differential equation?
- Is it **linear**?
- Does it have **constant coefficients**?
- What is the **order**?

Ordinary
An Ordinary Differential Equation or ODE has only one independent variable (for example, x, or t) and so uses ordinary derivatives. The alternative (equations for e.g. $f(x, y)$) is a partial differential equation: we saw some earlier but we will not solve them in this course.

Linearity
A differential equation is linear if every term in the equation contains none or exactly one of either the dependent variable or its derivatives. There are no products of the dependent variable with itself or its derivatives. Each term has at most one power of the equivalent of x or $\dot{x}$ or $\ddot{x}$ or $\ldots$; or $f(x)$ and its derivatives.
Examples:

$$f(x)\frac{\mathrm{d}f}{\mathrm{d}x} = -\omega^2 x \text{ is not linear} \qquad \frac{\mathrm{d}f}{\mathrm{d}x} = f^3(x) \text{ is not linear} \qquad \frac{\mathrm{d}^2f}{\mathrm{d}x^2} = -x^2f(x) + e^x \text{ is linear.}$$

Constant coefficients
A differential equation has constant coefficients if the dependent variable and all the derivatives are only multiplied by constants.
Examples: which have constant coefficients?

$$3\frac{\mathrm{d}f}{\mathrm{d}x} = -\omega^2 x\text{: yes} \qquad \frac{\mathrm{d}^2f}{\mathrm{d}x^2} = -x^2f(x) + e^x\text{: no} \qquad \frac{\mathrm{d}^2f}{\mathrm{d}x^2} + 3\frac{\mathrm{d}f}{\mathrm{d}x} + 2f(x) = \sin x e^x\text{: yes.}$$

Finally, a "trick" one:

$$3e^x\frac{\mathrm{d}f}{\mathrm{d}x} + e^xf(x) = x^3 \textbf{ does} \text{ have constant coefficients: divide the whole equation by } e^x.$$

Order

The order of a differential equation is the largest number of derivatives (of the dependent variable) ever taken.

Examples:

$f(x)\dfrac{\mathrm{d}f}{\mathrm{d}x} = -\omega^2 x$ is 1st order $\quad \dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2} = -x^2 f(x) + e^x$ is 2nd order $\quad \dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2} + 3\dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2}\dfrac{\mathrm{d}f}{\mathrm{d}x} = 0$ is 2nd order.

4.3 First order linear equations

First the general theory. A first order linear differential equation for $y(x)$ must be of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x).$$

If there is something multiplying the $\mathrm{d}y/\mathrm{d}x$ term, then divide the whole equation by this first.

Now suppose we calculate an **integrating factor**

$$I(x) = \exp\left(\int p(x)\,\mathrm{d}x\right).$$

Just this once, we won't bother about a constant of integration.

We multiply our equation by the integrating factor:

$$I(x)\frac{\mathrm{d}y}{\mathrm{d}x} + I(x)p(x)y = I(x)q(x).$$

and then observe that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(yI(x)\right) = \frac{\mathrm{d}y}{\mathrm{d}x}I(x) + y\frac{\mathrm{d}I}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x}I(x) + yp(x)I(x)$$

which is our left-hand-side. So we have the equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(yI(x)\right) = I(x)q(x)$$

which we can integrate (we hope):

$$yI(x) = \int I(x)q(x)\,\mathrm{d}x + C$$

$$y = \frac{1}{I(x)}\int I(x)q(x)\,\mathrm{d}x + \frac{C}{I(x)}.$$

The **last thing** we do is find C: we will do this using any initial conditions we've been given.

Example

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = 0 \qquad \text{and} \qquad y = 3 \ \text{ when } \ x = 0.$$

Here the integrating factor will be

$$I(x) = \exp\left(\int 2x\,\mathrm{d}x\right) = \exp x^2$$

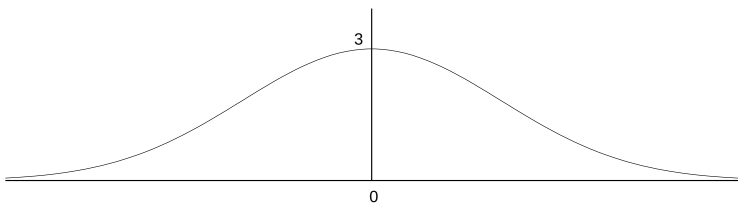
and our equation is

$$e^{x^2}\frac{\mathrm{d}y}{\mathrm{d}x} + 2xe^{x^2}y = 0.$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[ye^{x^2}\right] = 0 \qquad \Longrightarrow \qquad ye^{x^2} = C \qquad \Longrightarrow \qquad y = Ce^{-x^2}.$$

The last thing we do is use the initial conditions: at $x = 0$, $y = 3$ but our form gives at $x = 0$, $y = C$ so we need $C = 3$ and

$$y = 3e^{-x^2}.$$



Example

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \sin x \qquad \text{with } y(\pi/2) = 0.$$

First we need to get the equation into a form where the first term is just $\mathrm{d}y/\mathrm{d}x$: so divide by x:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x}y = \frac{\sin x}{x}.$$

Now we calculate the integrating factor:

$$I(x) = \exp\left(\int \frac{2}{x}\,\mathrm{d}x\right) = \exp\left(2\ln x\right) = \exp\ln\left(x^2\right) = x^2.$$

We multiply the whole system by x^2:

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = x\sin x$$

and now we can integrate:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2y) = x\sin x \qquad \Longrightarrow \qquad x^2y = \int x\sin x\,\mathrm{d}x + C$$

which we can integrate by parts:

$$x^2y = -x\cos x + \int \cos x \,\mathrm{d}x + C = -x\cos x + \sin x + C$$

so the general solution is

$$y = -\frac{\cos x}{x} + \frac{\sin x}{x^2} + \frac{C}{x^2}.$$

Finally, we use the initial condition $y = 0$ when $x = \pi/2$ to get

$$0 = -\frac{\cos(\pi/2)}{(\pi/2)} + \frac{\sin(\pi/2)}{(\pi/2)^2} + \frac{C}{(\pi/2)^2} = 0 + \frac{1}{(\pi/2)^2} + \frac{C}{(\pi/2)^2}.$$

which means $C = -1$ and our solution is

$$y = -\frac{\cos x}{x} - \frac{1 - \sin x}{x^2}.$$

Example

This time we will solve two different differential equations in parallel.

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = e^{-2x} \qquad \text{and} \qquad \frac{\mathrm{d}f}{\mathrm{d}x} + 3f = e^{-3x}$$

In this example, we don't actually have variable coefficients – but that just makes it easier!

$$\text{In both cases,} \quad I(x) = \exp\int 3\,\mathrm{d}x = e^{3x}.$$

$$\begin{aligned}
e^{3x}\frac{\mathrm{d}y}{\mathrm{d}x} + 3e^{3x}y = e^{x} &\qquad \text{and} \qquad e^{3x}\frac{\mathrm{d}f}{\mathrm{d}x} + 3e^{3x}f = 1.\\
\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{3x}y\right) = e^{x} &\qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{3x}f\right) = 1.\\
e^{3x}y = e^{x} + C_0 &\qquad \text{and} \qquad e^{3x}f = x + C_1.\\
y = e^{-2x} + C_0e^{-3x} &\qquad \text{and} \qquad f = xe^{-3x} + C_1e^{-3x}.
\end{aligned}$$

Notice that we haven't got any initial conditions so we can't determine the constants C_0 or C_1 here: what we have found is called the **general solution**.

4.4 Homogeneous linear equations.

A **homogeneous** linear equation is one in which all terms contain **exactly** one power of the dependent variable and its derivatives:

$$\text{e.g.} \qquad x^3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0.$$

For these equations, we can add up solutions: so if $f(x)$ is a solution and $g(x)$ is a solution:

$$x^3\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + 5x\frac{\mathrm{d}f}{\mathrm{d}x} + 2f = 0 \qquad \text{and} \qquad x^3\frac{\mathrm{d}^2 g}{\mathrm{d}x^2} + 5x\frac{\mathrm{d}g}{\mathrm{d}x} + 2g = 0$$

then so is $af(x) + bg(x)$ for any constants a and b:

$$x^3\frac{\mathrm{d}^2}{\mathrm{d}x^2}[af(x) + bg(x)] + 5x\frac{\mathrm{d}}{\mathrm{d}x}[af(x) + bg(x)] + 2[af(x) + bg(x)] = 0.$$

An nth order homogeneous linear equation will "always" (i.e. if it is well-behaved: don't worry about this detail) have exactly n independent solutions $y_1, \ldots, y_n$ and the general solution to the equation is

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n.$$

4.4.1 Special case: coefficients ax^r

Suppose we are given a differential equation in which the coefficient of the rth derivative is a constant multiple of x^r:

$$\text{e.g.} \quad x^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}y}{\mathrm{d}x} - 6y = 0.$$

Then if we try a solution of the form $y = x^m$ we get

$$y = x^m \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = mx^{m-1} \qquad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = m(m-1)x^{m-2}$$

and if we put this back into the original equation we get

$$x^2 m(m-1)x^{m-2} + 2mxx^{m-1} - 6x^m = 0$$

$$x^m(m(m-1) + 2m - 6) = 0 \qquad x^m(m^2 + m - 6) = 0.$$

Now x^m will take lots of values as x changes so we need

$$(m^2 + m - 6) = 0 \qquad \Longrightarrow \qquad (m-2)(m+3) = 0.$$

In this case we get two roots: $m_1 = 2$ and $m_2 = -3$. This means we have found two functions that work as solutions to our differential equation:

$$y_1 = x^{m_1} = x^2 \qquad \text{and} \qquad y_2 = x^{m_2} = x^{-3}.$$

But we know that if we have two solutions we can use any combination of them so our *general solution* is

$$y = c_1 x^2 + c_2 x^{-3}.$$

This works with an nth order ODE as long as the nth order polynomial for m has n different real roots.

Example

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6x\frac{\mathrm{d}y}{\mathrm{d}x} + 10y = 0.$$

Try $y = x^m$:

$$y = x^m \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = mx^{m-1} \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = m(m-1)x^{m-2}.$$

$$m(m-1)x^m - 6mx^m + 10x^m = 0 \implies x^m(m^2 - m - 6m + 10) = 0 \implies x^m(m-5)(m-2) = 0.$$

The general solution to this equation is

$$y = c_1x^5 + c_2x^2.$$

4.4.2 Special case: constant coefficients.

Now suppose we have a homogeneous equation with constant coefficients, like this one:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 0.$$

We try a solution $y = e^{\lambda x}$. This gives $\mathrm{d}y/\mathrm{d}x = \lambda e^{\lambda x}$ and $\mathrm{d}^2y/\mathrm{d}x^2 = \lambda^2 e^{\lambda x}$ so

$$\lambda^2 e^{\lambda x} + 5\lambda e^{\lambda x} + 6e^{\lambda x} = 0.$$

$$(\lambda^2 + 5\lambda + 6)e^{\lambda x} = 0 \qquad \text{for all } x.$$

Just like the polynomial case, the function of x will not be zero everywhere so we need

$$\lambda^2 + 5\lambda + 6 = 0 \qquad \implies \qquad (\lambda+2)(\lambda+3) = 0.$$

In this case we get two roots: $\lambda_1 = -2$ and $\lambda_2 = -3$. This means we have found two independent solutions:

$$y_1 = e^{\lambda_1 x} = e^{-2x} \qquad \text{and} \qquad y_2 = e^{\lambda_2 x} = e^{-3x},$$

and the *general solution* is

$$y = c_1e^{-2x} + c_2e^{-3x}.$$

Example

A third-order equation this time:

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - \frac{\mathrm{d}^2y}{\mathrm{d}x^3} - 2\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Trying $y = e^{\lambda x}$ gives

$$\lambda^3 - \lambda^2 - 2\lambda = 0 \qquad \implies \qquad \lambda(\lambda^2 - \lambda - 2) = 0 \qquad \implies \qquad \lambda(\lambda-2)(\lambda+1) = 0$$

which has three roots,

$$\lambda_1 = 0 \qquad \lambda_2 = 2 \qquad \lambda_3 = -1.$$

The general solution is

$$y = c_1e^{0x} + c_2e^{2x} + c_3e^{-x} = c_1 + c_2e^{2x} + c_3e^{-x}.$$

Notice that we have three constants here: in general we will always have N constants in the solution to an Nth order equation.

Example

Another second-order equation:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 0.$$

Trying $y = e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 5 = 0$$

which has two roots,

$$\lambda = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm 2i.$$

The general solution is then

$$y = Ae^{(-1+2i)x} + Be^{(-1-2i)x} = e^{-x}[Ae^{2ix} + Be^{-2ix}]$$

where A and B will be complex constants: but if y is real (which it usually is) then we can write the solution as

$$y = e^{-x}[c_1 \sin 2x + c_2 \cos 2x].$$

Repeated roots

If our polynomial for λ has two roots the same, then we will end up one short in our solution. The extra solution we need will be $xe^{\lambda x}$.

Example

Another third-order equation:

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Trying $y = e^{\lambda x}$ gives

$$\lambda^3 - 2\lambda^2 + \lambda = 0 \qquad \Longrightarrow \qquad \lambda(\lambda^2 - 2\lambda + 1) = 0 \qquad \Longrightarrow \qquad \lambda(\lambda-1)^2 = 0$$

which has only two distinct roots,

$$\lambda_1 = 0 \qquad \lambda_2 = \lambda_3 = 1.$$

The general solution is

$$y = c_1e^{0x} + c_2e^x + c_3xe^x = c_1 + c_2e^x + c_3xe^x.$$

4.5 Inhomogeneous linear equations.

What happens if there is a term with **none** of the dependent variable? That is, loosely, a term on the right hand side, or a function of x.

$$f_2(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + f_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + f_0(x)y = g(x).$$

In the most general case we can't do anything: but in one or two special cases we can.
If we already know the general solution to the homogeneous equation:

$$f_2(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + f_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + f_0(x)y = 0 \qquad \Longrightarrow \qquad y = c_1y_1(x) + c_2y_2(x)$$

then all we need is a particular solution to the whole equation: one function $u(x)$ that obeys

$$f_2(x)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + f_1(x)\frac{\mathrm{d}u}{\mathrm{d}x} + f_0(x)u = g(x).$$

Then the general solution to the whole equation is

$$y = c_1y_1(x) + c_2y_2(x) + u(x).$$

The solution to the homogeneous equation is called the complementary function or CF; the particular solution $u(x)$ is called the particular integral or PI. Finding it involves a certain amount of trial and error!

Special case: Coefficients x^r

In this case, we can only cope with one specific kind of RHS: a polynomial. We will see this by example:

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6x\frac{\mathrm{d}y}{\mathrm{d}x} + 10y = 6x^3.$$

The homogeneous equation in this case is one we've seen before:

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6x\frac{\mathrm{d}y}{\mathrm{d}x} + 10y = 0 \qquad \Longrightarrow \qquad y = c_1x^5 + c_2x^2.$$

Now as long as the power on the right is **not part of the CF** we can find the PI by trying a multiple of the right hand side:

$$y = Ax^3 \Longrightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3Ax^2 \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6Ax.$$

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6x\frac{\mathrm{d}y}{\mathrm{d}x} + 10y = x^2 6Ax - 6x3Ax^2 + 10Ax^3 = x^3[6A - 18A + 10A] = -2Ax^3$$

so for this to be a solution we need $-2A = 6$ so $A = -3$. Then the general solution to the full equation is

$$y = c_1x^5 + c_2x^2 - 3x^3.$$

A couple of words of warning about this kind of equation:

- If the polynomial for the power m has a repeated root then we fail
- If the polynomial for the power m has complex roots then we fail
- If a power on the RHS matches a power in the CF then we fail.

Special case: constant coefficients

Given a linear ODE with constant coefficients, we saw in the previous section that we can **always** find the general solution to the homogeneous equation (using $e^{\lambda x}$, $xe^{\lambda x}$ and so on), so we know how to find the CF. There are a set of standard functions to try for the PI, but that part is not guaranteed.

Example

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = e^{-x}.$$

First we need the **CF**. Try $y = e^{\lambda x}$ on the homogeneous equation:

$$\lambda^2 - 3\lambda + 2 = 0 \qquad \Longrightarrow \qquad (\lambda - 1)(\lambda - 2) = 0.$$

So there are two roots, $\lambda_1 = 1$ and $\lambda_2 = 2$. The CF is then

$$y_{\mathrm{CF}} = c_1 e^x + x_2 e^{2x}.$$

Next we need the **PI**. Since the RHS is e^{-x}, we try the form

$$y = Ae^{-x} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = -Ae^{-x} \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = Ae^{-x}.$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = Ae^{-x} + 3Ae^{-x} + 2Ae^{-x} = 6Ae^{-x}$$

so we need $A = 1/6$ for this to work. Our PI is

$$y_{\mathrm{PI}} = \frac{1}{6}e^{-x}$$

and the general solution is

$$y = c_1 e^x + x_2 e^{2x} + \frac{1}{6}e^{-x}.$$

Example

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = e^{-3x}.$$

This is only first-order: in fact we solved it in section 4.3 and the solution was

$$y = xe^{-3x} + C_1 e^{-3x}.$$

Let us solve it the way we have just learned. First the CF: try $y = e^{\lambda x}$ then

$$\lambda + 3 = 0$$

so $\lambda = -3$ and the CF is

$$y_{\mathrm{CF}} = C_1 e^{-3x}.$$

Now look for the PI. The RHS is e^{-3x} so our first thought might be to try Ae^{-3x}. But this is the CF: so we know when we try it we will get zero! So instead (motivated by knowing the answer in this case) we multiply by x and try

$$y = Axe^{-3x} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = Ae^{-3x} - 3Axe^{-3x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = Ae^{-3x} - 3Axe^{-3x} + 3Axe^{-3x} = Ae^{-3x}.$$

so we need $A = 1$ and we end up with the same solution we got last time.

In general, if the RHS matches the CF (or part of the CF) then we will multiply by x to get our trial function for the PI.

Example

This time we have initial conditions as well: remember we **always** use these as the very last thing we do.

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} = 2x \qquad \text{with } y = 3,\ \frac{\mathrm{d}y}{\mathrm{d}x} = -4 \text{ and } \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4 \text{ at } x = 0.$$

First we find the CF. Try $y = e^{\lambda x}$:

$$\lambda^3 + 2\lambda^2 + \lambda = 0 \implies \lambda(\lambda^2 + 2\lambda + 1) = 0 \implies \lambda(\lambda + 1)^2 = 0.$$

This has only two distinct roots: $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = -1$. Therefore the CF is:

$$y_{\mathrm{CF}} = c_1 + c_2e^{-x} + c_3xe^{-x}.$$

Now we look for the PI. The RHS is x so we try a function

$$y = Ax + B \implies \frac{\mathrm{d}y}{\mathrm{d}x} = A \implies \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 0.$$

This makes

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} = 0 + 0 + A$$

and no value of A can make this equal to x. What do we do when it fails?

- If the trial function fails, try multiplying by x.

[Note: in this case we could have predicted this because the B of our trial function is part of the CF.]

We want one more power of x so we try

$$y = Cx^2 + Ax \implies \frac{\mathrm{d}y}{\mathrm{d}x} = 2Cx + A \implies \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2C \text{ and } \frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 0.$$

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} = 0 + 4C + 2Cx + A$$

so we need

$$2Cx + 4C + A = 2x \qquad \text{which means } C = 1,\ A = -4.$$

Our general solution is

$$y = c_1 + c_2 e^{-x} + c_3 x e^{-x} + x^2 - 4x.$$

Now we apply the initial conditions:

$$\begin{aligned} y = c_1 + c_2 e^{-x} + c_3 x e^{-x} + x^2 - 4x \quad &\implies \quad y(0) = c_1 + c_2 = 3 \\ \frac{\mathrm{d}y}{\mathrm{d}x} = -c_2 e^{-x} + c_3 e^{-x} - c_3 x e^{-x} + 2x - 4 \quad &\implies \quad \frac{\mathrm{d}y}{\mathrm{d}x}(0) = -c_2 + c_3 - 4 = -4 \\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = c_2 e^{-x} - 2c_3 e^{-x} + c_3 x e^{-x} + 2 \quad &\implies \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}(0) = c_2 - 2c_3 + 2 = 4 \end{aligned}$$

The solution to this linear system is $c_2 = -2$, $c_3 = -2$, $c_1 = 5$ so our final answer is

$$y = 5 - 2e^{-x} - 2xe^{-x} + x^2 - 4x.$$

Table of functions to try for PI

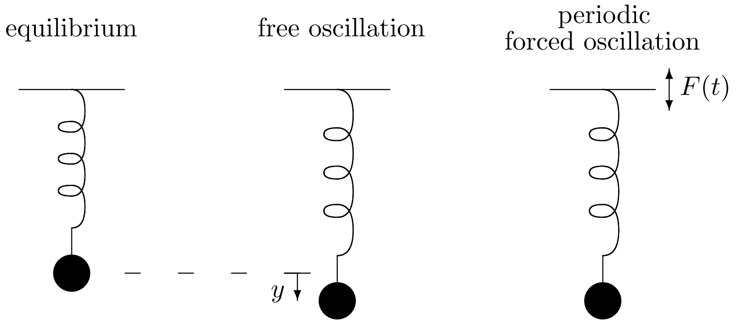
$f(x)$	Conditions on CF	First guess at PI
αe^{kx}	$\lambda = k$ not a root	$Ae^{\lambda x}$
$\sin kx$	$\lambda = ik$ not a root	$A\cos kx + B\sin kx$
$\cos kx$	$\lambda = ik$ not a root	$A\cos kx + B\sin kx$
x^n	$\lambda = 0$ not a root	$Ax^n + Bx^{n-1} + \cdots + C$

If the trial function given above matches with part of the CF then this won't work; instead we multiply by x (or x^2 if the result of that still matches the CF) and try again.

5 Fourier Series

5.1 Introduction: A model problem

Consider a forced spring:



The force exerted by the spring is a negative multiple of y: we will choose k so that the force is $-mk^2y$.

Unforced system

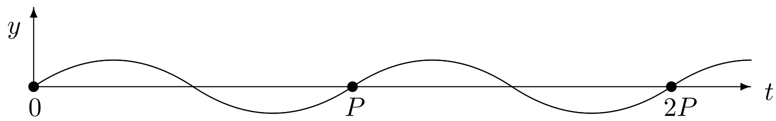
The governing equation is

$$\ddot{y} + k^2 y = 0$$

(recall the notation $\ddot{y} = \mathrm{d}^2y/\mathrm{d}t^2$) which has auxiliary equation $\lambda = \pm ik$ and general solution

$$y = a\cos\lambda t + b\sin\lambda t.$$

The solutions look like:



(this is $\sin kt$): they are periodic with period $P = 2\pi/k$. When $t = nP$ we have $\sin kt = \sin 2n\pi = 0$.

Add periodic forcing

The governing equation for the forced system is now

$$\ddot{y} + k^2 y = F(t).$$

So how do we find y? In other words, how will we find the PI?

5.2 Harmonic forcing

What we mean by **harmonic** forcing is that the forcing can be written as a pure sine wave $A\sin(\omega t - \delta)$ for some constants A and δ.

We will look here at a pure sine wave:

$$F(t) = \sin\omega t$$

which has period $T = 2\pi/\omega$. We **must** have $T \neq P$for the following discussion to work...

This system is easy to solve: try a particular solution $y = \alpha\cos\omega t + \beta\sin\omega t$. Then $\ddot{y} = -\omega^2 y$ and

$$\ddot{y} + k^2 y = F \implies -\omega^2 y + k^2 y = \sin\omega t \implies (k^2 - \omega^2)(\alpha\cos\omega t + \beta\sin\omega t) = \sin\omega t$$

Equating coefficients of $\cos\omega t$ and $\sin\omega t$:

$$\alpha = 0 \qquad \beta = \frac{1}{k^2 - \omega^2} \qquad y = \frac{1}{k^2 - \omega^2}\sin\omega t$$

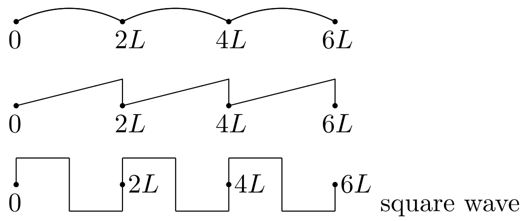
as long as $k \neq \omega$.

Notice that this solution will be very large if k is close to ω: this phenomenon is called **resonance**.

5.3 Periodic forcing

Now we look at the case where F has period (or 'repeat time') $2L$, but a non-harmonic shape:

e.g.



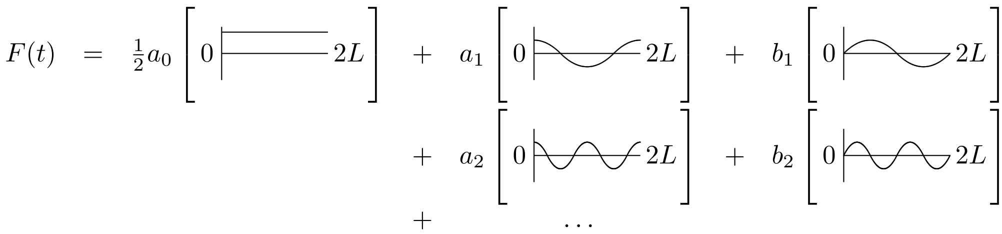
To solve

$$\ddot{y} + k^2 y = F(t),$$

we write F as a sum of harmonic waves:

$$\begin{aligned} F(t) &= \tfrac{1}{2}a_0 + a_1\cos\omega t + b_1 \sin\omega t \\ &\quad + a_2\cos 2\omega t + b_2\sin 2\omega t \\ &\quad + \dots \\ &= \tfrac{1}{2}a_0 + \textstyle\sum_{n=1}^{\infty} a_n \cos n\omega t + b_n \sin \omega t \end{aligned}$$

This series is called a **Fourier series**.



In our example case, we can solve the equation term by term:

$$\ddot{y}_n + k^2 y_n = a_n \cos n\omega t + b_n \sin n\omega t$$

We use a trial solution

$$\begin{aligned} y_n(t) &= \alpha_n \cos n\omega t + \beta_n \sin n\omega t \\ \ddot{y}_n &= -n^2\omega^2 y_n \\ \ddot{y}_n + k^2 y_n &= a_n \cos n\omega t + b_n \sin n\omega t = (k^2 - n^2\omega^2) y_n \end{aligned}$$

$$\implies \alpha_n = \frac{a_n}{k^2 - n^2\omega^2}, \qquad \beta_n = \frac{b_n}{k^2 - n^2\omega^2}$$

and we can build our complete solution from these **modes**:

$$y(t) = y_{\mathrm{CF}} + \tfrac{1}{2}y_0(t) + \sum_{n=1}^{\infty} y_n(t).$$

So the only part left is to calculate the a_n and b_n constants from $F(t)$: calculate the **Fourier series** of F.

5.4 Preparation for calculating a Fourier series: Orthogonality

We will need some properties of cos and sin.

$$\begin{array}{rrcl} \text{For } m \neq 0, & \displaystyle\int_0^{2L} \cos\left(\frac{\pi m x}{L}\right) \mathrm{d}x & = & \displaystyle\left[\frac{L}{m\pi}\sin\left(\frac{\pi m x}{L}\right)\right]_0^{2L} = 0 \\ \text{and} & \displaystyle\int_0^{2L} \sin\left(\frac{\pi m x}{L}\right) \mathrm{d}x & = & \displaystyle\left[\frac{-L}{m\pi}\cos\left(\frac{\pi m x}{L}\right)\right]_0^{2\pi} = -\frac{L}{m\pi} + \frac{L}{m\pi} = 0 \\ \text{For } m = 0, & \displaystyle\int_0^{2L} \cos\left(\frac{\pi m x}{L}\right) \mathrm{d}x & = & \displaystyle\int_0^{2L} 1\,\mathrm{d}x = 2L \\ \text{and} & \displaystyle\int_0^{2\pi} \sin\left(\frac{\pi m x}{L}\right) \mathrm{d}x & = & 0 \end{array}$$

Remember, there is nothing special about the name x in these integrals: the results are just as valid for the integrals over t or any other name.

$$\begin{aligned} \cos(a+b) &= \cos a \cos b - \sin a \sin b \\ \sin(a+b) &= \sin a \cos b + \cos a \sin b \end{aligned}$$

$$\begin{aligned} \cos a \cos b &= \tfrac{1}{2}(\cos(a+b) + \cos(a-b)) \\ \sin a \sin b &= \tfrac{1}{2}(\cos(a-b) - \cos(a+b)) \\ \sin a \cos b &= \tfrac{1}{2}(\sin(a+b) + \sin(a-b)) \end{aligned}$$

We can now integrate combinations. For all positive integers $m \geq 0$, $n \geq 0$:

$$\begin{aligned} \int_0^{2L} \sin\left(\frac{\pi m x}{L}\right)\sin\left(\frac{\pi n x}{L}\right) \mathrm{d}x &= \int_0^{2L} \frac{1}{2}\left\{\cos\left(\frac{\pi m x}{L} - \frac{\pi n x}{L}\right) - \cos\left(\frac{\pi m x}{L} + \frac{\pi n x}{L}\right)\right\} \mathrm{d}x \\ &= \frac{1}{2}\int_0^{2L} \cos\left(\frac{\pi(m-n)x}{L}\right) \mathrm{d}x - \frac{1}{2}\int_0^{2L} \cos\left(\frac{\pi(m+n)x}{L}\right) \mathrm{d}x \\ &= \frac{1}{2}\begin{cases} 2L & m = n \\ 0 & m \neq n \end{cases} - \frac{1}{2}\begin{cases} 2L & m = n = 0 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} L & m = n \neq 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Notice that one of the cases we have just shown is

$$\int_0^{2L} \sin\left(\frac{\pi m x}{L}\right)\sin\left(\frac{\pi n x}{L}\right) \mathrm{d}x = 0 \qquad \text{if } m \neq n.$$

We have shown that the functions $\sin(\pi k x/L)$, $k = 1, \ldots$ are **orthogonal** in the interval $0 \leq x \leq 2L$. (We're ignoring $k = 0$ as the function $\sin 0$ is not really a function.)

In exactly the same way, we can show that

$$\int_0^{2L} \cos\left(\frac{\pi m x}{L}\right)\cos\left(\frac{\pi n x}{L}\right) \mathrm{d}x = 0 \qquad \text{if } m \neq n,$$

so the functions $\cos(\pi kx/L)$, $k = 0, 1, \ldots$ are **orthogonal** in the interval $0 \le x \le 2L$; and

$$\int_0^{2L} \sin\left(\frac{\pi mx}{L}\right)\cos\left(\frac{\pi nx}{L}\right) \mathrm{d}x = 0.$$

So we have a set of mutually orthogonal functions in $0 \le x \le 2L$:

$$\cos\left(\frac{\pi mx}{L}\right) \quad \text{for } m = 0, 1, \ldots \qquad \sin\left(\frac{\pi nx}{L}\right) \quad \text{for } n = 1, 2, \ldots$$

This means **any** two different functions are orthogonal.

5.5 Calculating a Fourier series

We use the orthogonal property to calculate a Fourier series. We are given $F(x)$ in $0 \le x \le 2L$ (and F periodic, hence we know F everywhere) and we want

$$F(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi nx}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi nx}{L}\right).$$

The trick is to use the orthogonality. We pick one of our family of functions, multiply by it, and integrate: and the orthogonality condition means most of the terms on the right disappear. Let us do the sin terms first (this will give us b_n). We will multiply by $\sin(\pi mx/L)$, $m \ge 1$, and integrate:

$$\begin{aligned}
F(x) &= \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi nx}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi nx}{L}\right) \\
\int_0^{2L} \sin\left(\frac{\pi mx}{L}\right) F(x)\,\mathrm{d}x &= \tfrac{1}{2}a_0 \int_0^{2\pi} \sin\left(\frac{\pi mx}{L}\right)\mathrm{d}x + \sum_{n=1}^{\infty} a_n \int_0^{2\pi} \cos\left(\frac{\pi nx}{L}\right)\sin\left(\frac{\pi mx}{L}\right)\mathrm{d}x \\
&\quad + \sum_{n=1}^{\infty} b_n \int_0^{2\pi} \sin\left(\frac{\pi nx}{L}\right)\sin\left(\frac{\pi mx}{L}\right)\mathrm{d}x \\
&= 0 + 0 + \sum_{n=1}^{\infty} b_n \begin{cases} 0 & m \ne n \\ L & m = n \end{cases} = Lb_m. \\
b_m &= \frac{1}{L}\int_0^{2L} \sin\left(\frac{\pi mx}{L}\right) F(x)\,\mathrm{d}x \qquad \text{for } m \ge 1.
\end{aligned}$$

Recap

We were looking for the Fourier series for a function $F(x)$ with period $2L$:

$$F(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi nx}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi nx}{L}\right).$$

We showed the useful results:

$$\begin{aligned}
\int_0^{2L} \sin\left(\frac{\pi m x}{L}\right)\sin\left(\frac{\pi n x}{L}\right)\mathrm{d}x &= \begin{cases} L & m = n \neq 0 \\ 0 & \text{otherwise} \end{cases} \\
\int_0^{2L} \sin\left(\frac{\pi m x}{L}\right)\cos\left(\frac{\pi n x}{L}\right)\mathrm{d}x &= 0 \\
\int_0^{2L} \cos\left(\frac{\pi m x}{L}\right)\cos\left(\frac{\pi n x}{L}\right)\mathrm{d}x &= \begin{cases} 2L & m = n = 0 \\ L & m = n \neq 0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

and we have shown that

$$b_m = \frac{1}{L}\int_0^{2L} \sin\left(\frac{\pi m x}{L}\right)F(x)\,\mathrm{d}x \qquad \text{for } m \geq 1.$$

We can do a similar calculation, multiplying by $\cos(\pi m x/L)$, $m \geq 1$:

$$\begin{aligned}
F(x) &= \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{L}\right) \\
\int_0^{2L} \cos\left(\frac{\pi m x}{L}\right)F(x)\,\mathrm{d}x &= \tfrac{1}{2}a_0 \int_0^{2L} \cos\left(\frac{\pi m x}{L}\right)\mathrm{d}x + \sum_{n=1}^{\infty} a_n \int_0^{2L} \cos\left(\frac{\pi n x}{L}\right)\cos\left(\frac{\pi m x}{L}\right)\mathrm{d}x \\
&\quad + \sum_{n=1}^{\infty} b_n \int_0^{2L} \sin\left(\frac{\pi n x}{L}\right)\cos\left(\frac{\pi m x}{L}\right)\mathrm{d}x \\
&= 0 + \sum_{n=1}^{\infty} a_n \begin{cases} 0 & m \neq n \\ L & m = n \end{cases} + 0 = L a_m. \\
a_m &= \frac{1}{L}\int_0^{2L} \cos\left(\frac{\pi m x}{L}\right)F(x)\,\mathrm{d}x \qquad \text{for } m \geq 1.
\end{aligned}$$

and finally $\cos mx$ for $m = 0$, i.e. using the function 1:

$$\begin{aligned}
F(x) &= \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{L}\right) \\
\int_0^{2\pi} F(x)\,\mathrm{d}x &= \tfrac{1}{2}a_0 \int_0^{2L} 1\,\mathrm{d}x + \sum_{n=1}^{\infty} a_n \int_0^{2L} \cos\left(\frac{\pi n x}{L}\right)\mathrm{d}x + \sum_{n=1}^{\infty} b_n \int_0^{2L} \sin\left(\frac{\pi n x}{L}\right)\mathrm{d}x \\
&= \tfrac{1}{2}a_0(2L) + 0 + 0 = L a_0. \\
a_0 &= \frac{1}{L}\int_0^{2L} F(x)\,\mathrm{d}x = \frac{1}{L}\int_0^{2L} \cos\left(\frac{\pi 0 x}{L}\right)F(x)\,\mathrm{d}x
\end{aligned}$$

5.6 Odd and even functions

We have derived the formulae for the Fourier series. If

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\left\{a_n \cos\left(\frac{\pi n x}{L}\right) + b_n \sin\left(\frac{\pi n x}{L}\right)\right\}$$

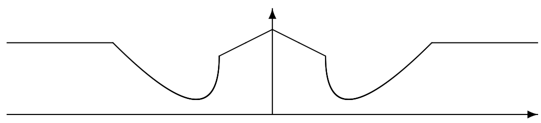
then

$$a_n = \frac{1}{L}\int_0^{2L} \cos\left(\frac{\pi n x}{L}\right) f(x)\,\mathrm{d}x \qquad\qquad b_n = \frac{1}{L}\int_0^{2L} \sin\left(\frac{\pi n x}{L}\right) f(x)\,\mathrm{d}x$$

But f is periodic with period $2L$, and so are all the cos and sin functions, so we could equally well have used the integration region $-L \le x \le L$, which is often more convenient:

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} \cos\left(\frac{\pi n x}{L}\right) f(x)\,\mathrm{d}x \\ b_n &= \frac{1}{L}\int_{-L}^{L} \sin\left(\frac{\pi n x}{L}\right) f(x)\,\mathrm{d}x \end{aligned}$$

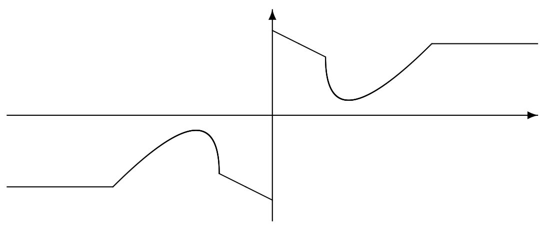
Now let's look at the behaviour of f between $-L$ and L. If $f(-x) = f(x)$ for every value of x in the range, then f is said to be an **even** function:



You can see that if we integrate this over the whole (symmetric) range, we get double the integral over the right half:

$$\int_{-L}^{L} f_{\text{even}}\,\mathrm{d}x = 2\int_0^{L} f_{\text{even}}\,\mathrm{d}x.$$

If, on the other hand, $f(-x) = -f(x)$ for every value of x in the range, then f is said to be an **odd** function:



Integrating this over the symmetric range (and remembering that areas above a negative curve count negative) will give zero:

$$\int_{-L}^{L} f_{\text{odd}}\,\mathrm{d}x = 0.$$

Products of odd and even functions are odd or even too following these rules:

$$\begin{aligned} \text{Even function} \times \text{Even function} &= \text{Even function} \\ \text{Even function} \times \text{Odd function} &= \text{Odd function} \\ \text{Odd function} \times \text{Odd function} &= \text{Even function} \end{aligned}$$

Cosine (cos) is an even function; sin is an odd function.

Even functions

If $f(x)$ is even then $f(x)\cos(n\pi x/L)$ is even and $f(x)\sin(n\pi x/L)$ is odd. The integrals in the standard form become:

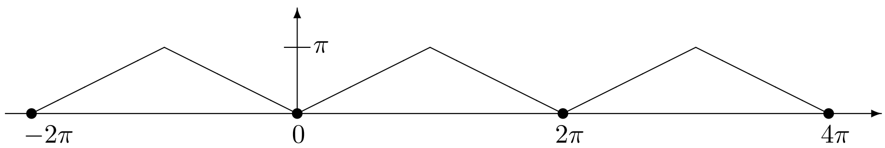
$$a_n = \frac{2}{L}\int_0^L \cos\left(\frac{\pi n x}{L}\right) f(x)\,\mathrm{d}x, \qquad b_n = 0.$$

Odd functions

If $f(x)$ is odd then $f(x)\cos(n\pi x/L)$ is odd and $f(x)\sin(n\pi x/L)$ is even. The integrals in the standard form become:

$$a_n = 0, \qquad b_n = \frac{2}{L}\int_0^L \sin\left(\frac{\pi n x}{L}\right) f(x)\,\mathrm{d}x.$$

5.7 Fourier series example: sawtooth function



This function is periodic with period 2π, so we will put $L = \pi$.

$$f(x) = \begin{cases} x & 0 \le x \le \pi \\ 2\pi - x & \pi \le x \le 2\pi \end{cases} \qquad \text{or} \qquad f(x) = \begin{cases} -x & -\pi \le x \le 0 \\ x & 0 \le x \le \pi \end{cases}$$

Now look at the form of $f(x)$ between $-\pi$ and π. It is clear from the graph that $f(-x) = f(x)$ so this is an **even** function.

We will use the coefficient formulae we derived in the last section: if

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\left\{a_n \cos\left(\frac{\pi n x}{L}\right) + b_n \sin\left(\frac{\pi n x}{L}\right)\right\}$$

and $f(x)$ is even, then

$$a_n = \frac{2}{L}\int_0^L \cos\left(\frac{\pi n x}{L}\right) f(x)\,\mathrm{d}x \qquad b_n = 0$$

When we put in $L = \pi$ this becomes much simpler:

$$a_n = \frac{2}{\pi}\int_0^{\pi} \cos nx\, f(x)\,\mathrm{d}x \qquad b_n = 0$$

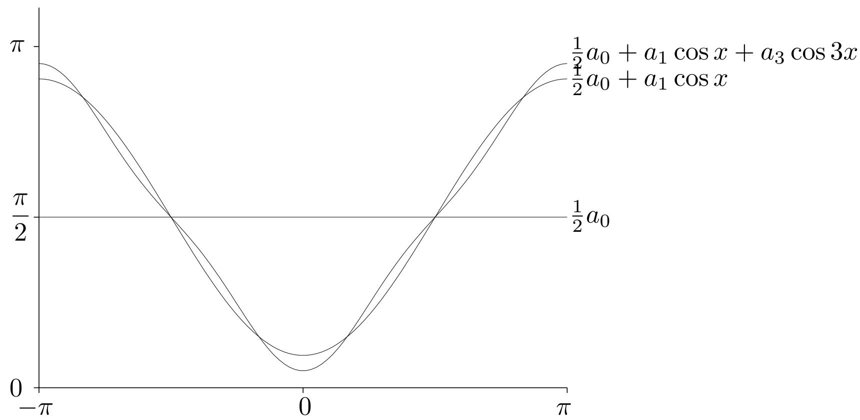
In this case we have

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} x\,\mathrm{d}x = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{2}{\pi}\frac{\pi^2}{2} = \pi$$

$$
\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^\pi \cos nx f(x)\,\mathrm{d}x = \frac{2}{\pi}\int_0^\pi x\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\left(\left[x\frac{\sin nx}{n}\right]_0^\pi - \int_0^\pi \frac{\sin nx}{n}\,\mathrm{d}x\right) \quad \text{by parts} \\
&= \frac{2}{\pi}\left(0 - \left[-\frac{\cos nx}{n^2}\right]_0^\pi\right) = \frac{2}{\pi}\frac{1}{n^2}(\cos n\pi - 1) = \frac{2}{n^2\pi}\begin{cases} 0 & n \text{ even} \\ -2 & n \text{ odd.}\end{cases}
\end{aligned}
$$

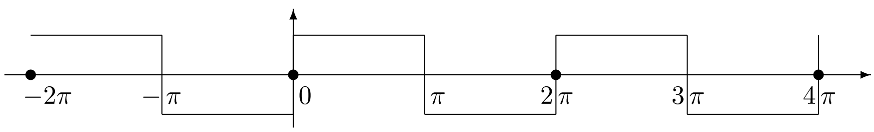
So the Fourier series for this sawtooth function is

$$
\begin{aligned}
f(x) &= \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}(\cos n\pi - 1)\cos nx \\
&= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n\ \text{odd}}^{\infty} \frac{1}{n^2}\cos nx \\
&= \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right).
\end{aligned}
$$



The coefficients in this case **decrease** like $1/n^2$: this sort of decay occurs when f is **continuous** (i.e. has no jumps). We may only need a few terms to get a good approximation to the shape: the graph above shows only three terms of the series.

5.8 Example Fourier series with discontinuities: Square wave



$$
f(x) = \begin{cases} -1 & -\pi < x \le 0 \\ 1 & 0 < x \le \pi \end{cases}
$$

This function is periodic with period 2π, but it is not continuous: there are jumps at 0, π, 2π, 3π, ...

This time, the function is **odd**: the values on the left are negative the values on the right.
Again, we use the formulae we derived: if

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{\pi n x}{L}\right) + b_n \sin\left(\frac{\pi n x}{L}\right) \right\}$$

then

$$a_n = 0, \qquad b_n = \frac{2}{L}\int_0^L \sin\left(\frac{\pi n x}{L}\right) f(x)\,\mathrm{d}x$$

and for the special case $L = \pi$ this becomes

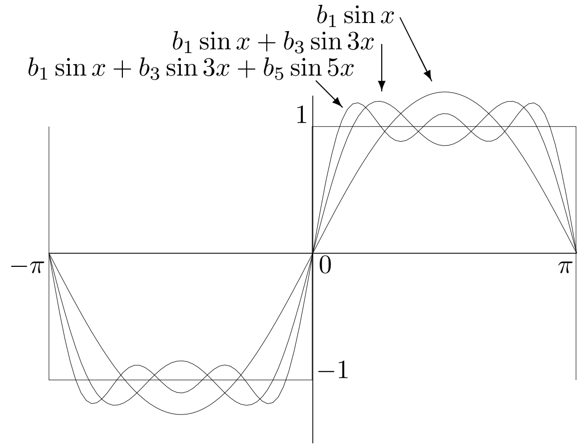
$$a_n = 0, \qquad b_n = \frac{2}{\pi}\int_0^\pi \sin nx f(x)\,\mathrm{d}x.$$

$$b_n = \frac{2}{\pi}\int_0^\pi \sin nx(1)\,\mathrm{d}x = \frac{2}{\pi}\left[-\frac{\cos nx}{n}\right]_0^\pi = \frac{2}{n\pi}(1-\cos n\pi) = \frac{2}{n\pi}\begin{cases} 2 & n \text{ odd} \\ 0 & n \text{ even.}\end{cases}$$

So the Fourier series for the square wave is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = \sum_{n \text{ odd}} \frac{4}{n\pi}\sin nx = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$

Here the coefficients decrease like $1/n$: this happens when f is discontinuous (has a jump). The errors are more visible after a few terms than for the continuous case:



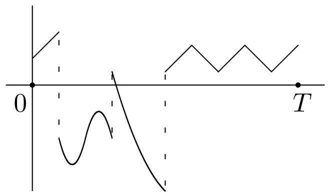
5.9 Will the Fourier series converge?

Dirichlet conditions: sufficient, but not necessary, for convergence:

- $f(x)$ is defined and single-values, except possily at a finite number of points in the periodic range
- $f(x)$ is periodic

- $f(x)$ and $\mathrm{d}f/\mathrm{d}x$ are piecewise continuous.

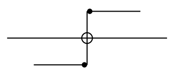
e.g.



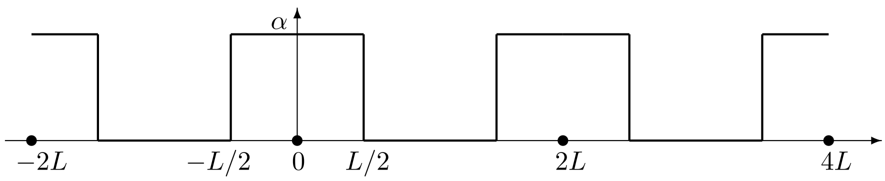
is fine, as long as it is periodic.

Where f is continuous (no jump), the Fourier series converges to $f(x)$.
Where f has a jump, the Fourier series converges to the middle of the gap. For example, with the square wave:

$$\begin{aligned}\sum b_n \sin nx &= 0 \quad \text{at } x = 0 \text{ (and } x = \pi,\ 2\pi,\ \ldots) \\ &= \tfrac{1}{2}(1 + (-1)) = \tfrac{1}{2}\{f(0_+) + f(0_-)\}.\end{aligned}$$



5.10 Example with a different period: another square wave



$$F(x) = \begin{cases} \alpha & -L/2 < x < L/2 \\ 0 & L/2 < x < 3L/2 \end{cases}$$

Note that F is an **even** function. This means we are expecting a cosine series, because cos is even and sin is odd. The coefficients will be

$$a_n = \frac{2}{L}\int_0^L \cos\left(\frac{\pi n x}{L}\right) F(x)\,\mathrm{d}x \qquad\qquad b_n = 0$$

$$\begin{aligned}a_0 &= \frac{2}{L}\int_0^L F(x)\,\mathrm{d}x = \frac{2}{L}\int_0^{L/2} \alpha\,\mathrm{d}x = \alpha \qquad \text{twice the average value of } F \\ a_n &= \frac{2}{L}\int_0^L F(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \frac{2\alpha}{L}\int_0^{L/2} \cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\ &= \frac{2\alpha}{L}\left[\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]_0^{L/2} = \frac{2\alpha}{n\pi}\sin\left(\frac{n\pi}{2}\right)\end{aligned}$$

We can now construct the series for $F(x)$:

$$\begin{aligned} F(x) &= \frac{\alpha}{2} + \sum_{n=1}^{\infty} \frac{2\alpha}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos\left(\frac{n\pi x}{L}\right) \\ &= \frac{\alpha}{2} + \frac{2\alpha}{\pi} \left\{ \cos\left(\frac{\pi x}{L}\right) - \frac{1}{3}\cos\left(\frac{3\pi x}{L}\right) + \frac{1}{5}\cos\left(\frac{5\pi x}{L}\right) - \frac{1}{7}\cos\left(\frac{7\pi x}{L}\right) + \cdots \right\} \end{aligned}$$

Note: this is a cosine series because $F(x)$ is **even**. The terms decrease like $1/n$ for large n because F has discontinuities.

Let's look at the discontinuity point $L/2$:

$$\begin{aligned} F(L/2) &= \frac{\alpha}{2} + \sum_{n \text{ odd}} \frac{2\alpha}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos\left(\frac{n\pi L}{2L}\right) \\ & \text{and } \cos\left(\frac{n\pi}{2}\right) = 0 \text{ when } n \text{ is odd} \\ F(L/2) &= \frac{\alpha}{2} \end{aligned}$$

so it converges to the middle of the gap as we expected.

One more case to look at: at $x = 0$, $F(x) = \alpha$ from the definition, so

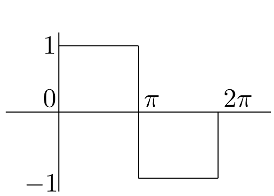
$$\alpha = \frac{\alpha}{2} + \frac{2\alpha}{\pi} \left\{ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \right\}$$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi/4}{.}$$

5.11 Superposition of Fourier series

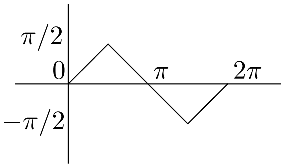
When we have a periodic function that is made out of two simpler functions, we can exploit this. I will explain by example.

Square wave: $f(x)$



$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, \mathrm{d}x \qquad (\text{odd function, } a_n = 0) \\ &= \frac{2}{\pi} \int_0^{\pi} 1. \sin nx \, \mathrm{d}x = \frac{2}{\pi} \left[-\frac{\cos nx}{n} \right]_0^{\pi} \\ &= \frac{2}{n\pi} (1 - \cos n\pi) \end{aligned}$$

'Triangular' wave: $g(x)$

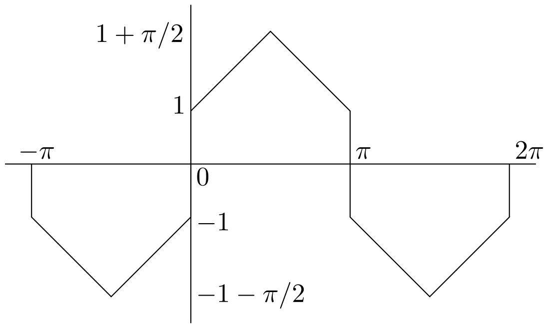


$$
\begin{aligned}
B_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin nx\,\mathrm{d}x \qquad \text{(odd function, } A_n = 0\text{)}\\
&= \frac{2}{\pi}\int_{0}^{\pi} g(x)\sin nx\,\mathrm{d}x\\
&= \frac{2}{\pi}\int_{0}^{\pi/2} x\sin nx\,\mathrm{d}x + \frac{2}{\pi}\int_{\pi/2}^{\pi}(\pi - x)\sin nx\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2}{\pi}\left\{\left[x\frac{(-\cos nx)}{n}\right]_0^{\pi/2} - \int_0^{\pi/2}\frac{-\cos nx}{n}\,\mathrm{d}x + \left[(\pi - x)\frac{(-\cos nx)}{n}\right]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi}(-1)\frac{(-\cos nx)}{n}\,\mathrm{d}x\right\}\\
&= \frac{2}{\pi}\left\{-\frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right) - 0 + \int_0^{\pi/2}\frac{\cos nx}{n}\,\mathrm{d}x + 0 + \frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right) - \int_{\pi/2}^{\pi}\frac{\cos nx}{n}\,\mathrm{d}x\right\}\\
&= \frac{2}{\pi}\left\{\left[\frac{\sin nx}{n^2}\right]_0^{\pi/2} - \left[\frac{\sin nx}{n^2}\right]_{\pi/2}^{\pi}\right\} = \frac{2}{\pi}\left\{\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\right\} = \frac{4}{n^2\pi}\sin\left(\frac{n\pi}{2}\right)
\end{aligned}
$$

Now we can add these two functions together to get a new function, also periodic with period 2π and odd:

$$h(x) = f(x) + g(x)$$



Then

$$
\begin{aligned}
h(x) &= \sum_1^{\infty} b_n \sin nx + \sum_1^{\infty} B_n \sin nx\\
&= \sum_1^{\infty}(b_n + B_n)\sin nx, \qquad \text{which gives us the Fourier series of } h\text{: a sine series.}
\end{aligned}
$$

A few of the coefficients:

$$\begin{array}{lccccc}
b_n + B_n & = & \frac{2}{n\pi}(1-\cos n\pi) + \frac{4}{n^2\pi}\sin\left(\frac{n\pi}{2}\right) & & & \\
n=1 & \frac{4}{\pi} & + & \frac{4}{\pi} & = & \frac{8}{\pi} \\
n=2 & 0 & + & 0 & = & 0 \\
n=3 & \frac{4}{3\pi} & - & \frac{4}{9\pi} & = & \frac{8}{9\pi} \\
n=4 & 0 & + & 0 & = & 0 \\
n=5 & \frac{4}{5\pi} & + & \frac{4}{25\pi} & = & \frac{24}{25\pi}
\end{array}$$

Note that we can have $1/n$ and $1/n^2$ type behaviour mixed together in the coefficients. There will always be some of the $1/n$ type if the function has discontinuities (jumps): we can see it here because of the jump in h at $x = 0, \pi, \ldots$

5.12 Integration of a Fourier series

The Fourier series for $f(x)$ can be integrated term by term provided that $f(x)$ is piecewise continuous in the period $2L$ (i.e. only a finite number of jumps):

$$\int_\alpha^\beta f(x)\,\mathrm{d}x = \int_\alpha^\beta \tfrac{1}{2}a_0\,\mathrm{d}x + \sum_{n=1}^{\infty} a_n \int_\alpha^\beta \cos\frac{n\pi x}{L}\,\mathrm{d}x + b_n \int_\alpha^\beta \sin\frac{n\pi x}{L}\,\mathrm{d}x.$$

5.13 Fourier series – Parseval's identity

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$$

is the standard Fourier series for a function with period $2L$.

Now consider

$$\begin{aligned}
\int_0^{2L} f(x)f(x)\,\mathrm{d}x &= \int_0^{2L}\left\{\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)\right\} f(x)\,\mathrm{d}x \\
&= \tfrac{1}{2}a_0\int_0^{2L} f(x)\,\mathrm{d}x + \sum_{n=1}^{\infty}\left(a_n\int_0^{2L}\cos\frac{n\pi x}{L}f(x)\,\mathrm{d}x + b_n\int_0^{2L}\sin\frac{n\pi x}{L}f(x)\,\mathrm{d}x\right)
\end{aligned}$$

where we are allowed to swap the order of the integral and the infinite sum because the Fourier series converges uniformly: don't worry about the definition of this, but do remember we can't always switch the order like this!

$$= \tfrac{1}{2}a_0La_0 + \sum_{n=1}^{\infty}(a_nLa_n + b_nLb_n) = L\left[\tfrac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2+b_n^2)\right].$$

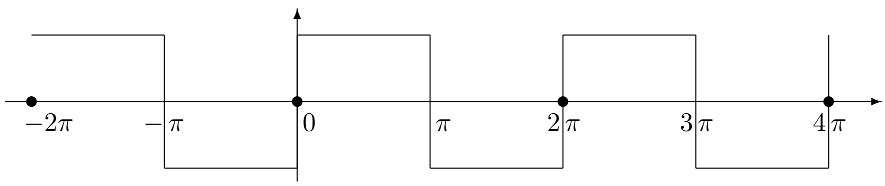
To put it another way,

$$\frac{1}{L}\int_0^{2L} f(x)f(x)\,\mathrm{d}x = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

This is **Parseval's identity**.

Example

Remember the square wave, of height 1 and period 2π:



The Fourier series for this function was

$$f(x) = \sum_1^{\infty} b_n \sin nx \qquad \text{with} \qquad b_n = \begin{cases} \dfrac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

Parseval's identity gives

$$\begin{aligned} \frac{1}{\pi}\int_0^{2\pi} f^2(x)\,\mathrm{d}x &= \sum_{n=1}^{\infty} b_n^2 \\ 2 &= \frac{16}{\pi^2}\left[1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots\right] \end{aligned}$$

which tells us that

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots = \frac{\pi^2}{8}.$$

6 Linear Equations

6.1 Notation

We begin with a review of material from B3C. First some notation:

Vectors

$$\underline{v} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}, \quad \underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}.$$

If the vector has two components we call it a **two-dimensional vector**, three components make it a **three-dimensional vector** and so on.

Linear combinations

A linear combination of variables is an expression of the form

$$3x_1 + 2x_2, \qquad c_1 x + c_2 y + c_3 z$$

where c_j $(j = 1, 2, 3)$ are constants.

In the same way we can write a linear combination of vectors:

$$4\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 3\begin{pmatrix} 6 \\ 4 \end{pmatrix}; \qquad \lambda\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \mu\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

where λ, μ are constants.

Linear equation Examples of linear equations are:

$$x + y - 2z = 6 \ \text{ (a plane in 3D space)}$$

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 = A$$

with a_1, a_2, a_3, a_4, A constant.

Set of linear equations An example set of linear equations could be

$$\begin{array}{ccccc} x & - & y & = & 2 \\ 3x & + & y & = & 4 \end{array}$$

or more generally,

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1N}x_N & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2N}x_N & = & b_2 \\ \vdots & & & & \ddots & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mN}x_N & = & b_m \end{array}$$

This is a set of m linear equations in N unknowns $x_1 \ldots x_N$, with constant coefficients $a_{11} \ldots a_{mN}$, $b_1 \ldots b_m$.

Matrix notation The sets of linear equations above can be written in matrix notation as

$$\begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

for the first, and

$$\underline{\underline{A}}\,\underline{x} = \underline{b}$$

for the second, where

$$\underline{\underline{A}} = \begin{pmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mN} \end{pmatrix}$$

is a matrix of constant coefficients,

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \text{ is to be found, and } \underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \text{ is a constant vector.}$$

6.2 Echelon form

One way to solve a set of linear equations is by reduction to row-echelon form: using row operations:

- multiply a row by any constant (i.e. a number)
- interchange two rows
- add a multiple of one row to another

Row-echelon form means:

- any **all-zero** rows are at the bottom of the reduced matrix
- in a non-zero row, the first-from-left nonzero value is 1
- the first '1' in each row is to the right of the first '1' in the row above

All the operations can be carried out on the augmented matrix

$$\left(\underline{\underline{A}} \,\middle|\, \underline{b} \right).$$

Once the process is complete, it's easy to find the solution to the set of equations by back-substitution.

Example

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & 3x_3 & = & 9 \\ 2x_1 & - & x_2 & + & x_3 & = & 8 \\ 3x_1 & & & - & x_3 & = & 3 \end{array}$$

$$\left(\begin{array}{rrr|r} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right) \to \begin{array}{c} R_1 \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \left(\begin{array}{rrr|r} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & -6 & -10 & -24 \end{array}\right)$$

$$\to \begin{array}{c} R_1 \\ -R_2/5 \\ -R_3/2 \end{array} \left(\begin{array}{rrr|r} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 3 & 5 & 12 \end{array}\right)$$

$$\to \begin{array}{c} R_1 \\ R_2 \\ R_3 - 3R_2 \end{array} \left(\begin{array}{rrr|r} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 6 \end{array}\right)$$

$$\to \begin{array}{c} R_1 \\ R_2 \\ R_3/2 \end{array} \left(\begin{array}{rrr|r} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Now we can solve from the bottom up:

$$\begin{array}{rcl} x_3 & = & 3 \\ x_2 & = & 2 - x_3 = -1 \\ x_1 & = & 9 - 2x_2 - 3x_3 = 2. \end{array}$$

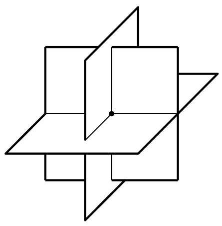
In this case we had three unknowns and three non-zero rows in echelon form, giving a unique solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}.$$

The **rank**, R, is the number of non-zero rows in echelon form. The ranks of the matrix and the augmented matrix determine the type of the solution.

Unique solution

If we have N variables and N nonzero rows in echelon form, we always get a unique solution.
In three dimensions this means geometrically that three independent planes meet at a point.



Zero rows

What if row reduction leads to, for example,

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)?$$

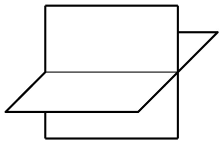
The rank R of the augmented matrix is 2, the number of variables $N = 3$.
In this situation x_3 can have any value: so to solve, put $x_3 = \lambda$, say, then

$$\begin{aligned} x_2 &= 2 - x_3 = 2 - \lambda \\ x_1 &= 5 - 2x_2 = 5 - 2(2 - \lambda) = 1 + 2\lambda \end{aligned}$$

so the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

In three dimensions with two independent rows this means geometrically that two independent planes intersect in a line.



Here λ is a free parameter. In general, we expect $N - R$ free parameters in the solution.

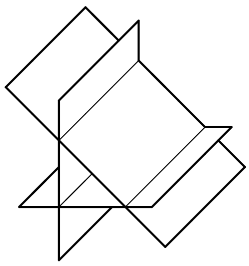
No solution

What if row reduction leads to, for example,

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 0 & 1 & 6 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right) ?$$

Here the rank R of the augmented matrix exceeds the rank of the left-hand-side. There is no value of x_3 we can choose that will give a solution.

Geometrically, this corresponds to a situation where three planes which are not independent do not intersect at all:



6.3 Relationship to eigenvalues and eigenvectors

Remember from B3C: for an $N \times N$ matrix $\underline{\underline{A}}$, if

$$\underline{\underline{A}}\,\underline{v} = \lambda \underline{v}$$

with $\underline{v} \neq \underline{0}$ then $\underline{v}$ is an **eigenvector** of $\underline{\underline{A}}$ with **eigenvalue** λ.

(Any multiple of $\underline{v}$ is also an eigenvector with eigenvalue λ: just pick some convenient form.)

The equation above can be written as

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{v} = \underline{0}$$

which is a matrix-vector equation with augmented matrix

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}} | \underline{0}).$$

If the rank is the same as the number of variables, $R = N$, then there is a unique solution. But

$$\underline{v} = \underline{0}$$

is a solution (the *trivial* solution) so to get another solution we need $R < N$. This means that when matrix $\underline{\underline{A}} - \lambda \underline{\underline{I}}$ is reduced to echelon form there must be a zero row: or (equivalently) the determinant of matrix $\underline{\underline{A}} - \lambda \underline{\underline{I}}$ is zero.

The determinant $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}})$ is a polynomial in λ of degree N, so there are at most N different eigenvalues. A useful fact (not proved here) is $\det(\underline{\underline{A}}) = \lambda_1 \lambda_2 \cdots \lambda_N$.

Example

$$\underline{\underline{A}} = \begin{pmatrix} 5 & -2 \\ 9 & -6 \end{pmatrix}.$$

$$\begin{aligned} |\underline{\underline{A}} - \lambda \underline{\underline{I}}| &= \begin{vmatrix} 5-\lambda & -2 \\ 9 & -6-\lambda \end{vmatrix} = (5-\lambda)(-6-\lambda) - (-2)(9) \\ &= (-30 + \lambda + \lambda^2) + 18 = \lambda^2 + \lambda - 12 = (\lambda+4)(\lambda-3) \end{aligned}$$

so the matrix has eigenvalues $\lambda_1 = 3$, $\lambda_2 = -4$.

$\lambda_1 = 3$:

$$(\underline{\underline{A}} - \lambda_1 \underline{\underline{I}})\underline{v}_1 = \underline{0} \text{ and } \underline{v}_1 = \begin{pmatrix} a \\ b \end{pmatrix} \implies \begin{pmatrix} 2 & -2 \\ 9 & -9 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$a - b = 0 \qquad a = b \qquad \underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ (or any multiple of this).}$$

Check:

$$\underline{\underline{A}}\,\underline{v}_1 = \begin{pmatrix} 5 & -2 \\ 9 & -6 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda_1 \underline{v}_1.$$

$\lambda_2 = -4$:

$$(\underline{\underline{A}} - \lambda_2 \underline{\underline{I}})\underline{v}_2 = \underline{0} \text{ and } \underline{v}_2 = \begin{pmatrix} a \\ b \end{pmatrix} \implies \begin{pmatrix} 9 & -2 \\ 9 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$9a - 2b = 0 \qquad 9a = 2b \qquad \underline{v}_2 = \begin{pmatrix} 2 \\ 9 \end{pmatrix} \text{ (or any multiple of this).}$$

Check:

$$\underline{\underline{A}}\,\underline{v}_2 = \begin{pmatrix} 5 & -2 \\ 9 & -6 \end{pmatrix} \begin{pmatrix} 2 \\ 9 \end{pmatrix} = \begin{pmatrix} -8 \\ -36 \end{pmatrix} = -4 \begin{pmatrix} 2 \\ 9 \end{pmatrix} = \lambda_2 \underline{v}_2.$$

Properties of eigenvalues and eigenvectors

- Eigenvectors for different eigenvalues are linearly independent.
- There may be multiple eigenvalues with the same value.
- $\lambda = 0$ can be an eigenvalue but $\underline{v} = \underline{0}$ can't be an eigenvector.
- λ can be complex.

6.4 Linear (in)dependence of vectors

- Vectors are linearly dependent if there is some non-trivial linear combination of them that sums to zero.
- Vectors are linearly independent if there is no non-trivial linear combination of them that sums to zero.

Example:

$$\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \underline{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

Here $\underline{v}_1 + \underline{v}_2 - \underline{v}_3 = \underline{0}$ so the vectors are linearly dependent.
Given vectors $\underline{v}_1, \cdots, \underline{v}_N$, they are independent if the linear equation

$$\alpha_1 \underline{v}_1 + \cdots + \alpha_N \underline{v}_N = \underline{0}$$

has $\alpha_1 = \cdots = \alpha_N = 0$ as the **only** solution. (Note: this is the trivial solution, so this means there is no non-trivial solution.)
This linear equation of vectors is a set of linear equations of scalars. For example, using the vectors above, we can write it as

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

in which the vectors have become the matrix columns.
We can solve this by reduction to echelon form: in this case we get

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

which gives

$$\begin{aligned} \alpha_3 &= \lambda \\ \alpha_2 &= -\alpha_3 = -\lambda \\ \alpha_1 &= -2\alpha_3 - \alpha_2 = -\lambda \end{aligned}$$

The non-trivial solution to the equation for the coefficients α is

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \lambda \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

so as we found above, the set of vectors is linearly dependent.

6.4.1 Write a vector as a sum of other given vectors

Suppose we are given vectors $\underline{v}_1$, $\underline{v}_2$, $\underline{v}_3$, and another vector $\underline{w}$, and asked to find coefficients α_1, α_2, α_3 such that $\underline{w} = \alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 + \alpha_3 \underline{v}_3$. This can be solved using the same methods.

Example: Write $\begin{pmatrix} 9 \\ 8 \\ 3 \end{pmatrix}$ as a combination of $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$.

This is just a set of linear equations:

$$\alpha_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 9 \\ 8 \\ 3 \end{pmatrix}$$

which we can write as

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 9 \\ 8 \\ 3 \end{pmatrix}$$

and solve using, for example, row reduction to echelon form.

We did this case earlier, finding

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}.$$

If there are N vectors $\underline{v}_n$ that are linearly independent with N dimensions, then

- every vector $\underline{w}$ can be written as a combination of the vectors $\underline{v}_1, \cdots, \underline{v}_N$: this means that the set of vectors $\underline{v}_1, \cdots, \underline{v}_N$ **spans** the N-dimensional space
- for any $\underline{w}$, there is a unique solution for the N coefficients $\alpha_1, \cdots, \alpha_N$ to write $\underline{w}$ in terms of $\underline{v}_1, \cdots, \underline{v}_N$.

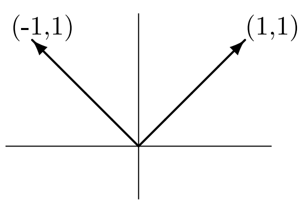
The N vectors $\underline{v}_1, \cdots, \underline{v}_N$ are said to be a **basis** for the N-dimensional space.

In general, a set of linearly independent vectors will span some subspace (defined as the set of all vectors that can be made as linear combinations of them) and they will be a basis for that space.

6.5 Orthonormal sets of vectors

6.5.1 Orthogonal vectors

Orthogonal vectors are vectors at right angles to each other:



$$(-1, 1) \cdot (1, 1) = -1 + 1 = 0.$$

Vectors $\underline{v}_1$ and $\underline{v}_2$ are orthogonal if $\underline{v}_1 \cdot \underline{v}_2 = 0$.
A set of vectors $\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_N$ is (mutually) orthogonal if

$$\underline{v}_i \cdot \underline{v}_j = 0 \quad \text{for all} \quad i \neq j, \quad i = 1 \ldots N, \; j = 1 \ldots N.$$

Recap: Orthogonal vectors

A set of vectors $\underline{v}_1$, $\underline{v}_2$, $\cdots \underline{v}_N$ is (mutually) orthogonal if

$$\underline{v}_i \cdot \underline{v}_j = 0 \quad \text{for all} \quad i \neq j, \quad i = 1 \ldots N, \; j = 1 \ldots N.$$

Examples

$$\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \text{ and } \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \text{ are orthogonal.}$$

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \text{ are orthogonal.}$$

$$\begin{pmatrix} x \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ y \end{pmatrix} \text{ are orthogonal.}$$

6.5.2 Normalised vectors

A vector is **normal** or a **unit vector** if it has magnitude 1.
If $\underline{u} \cdot \underline{u} = 1$ then the vector $\underline{u}$ is normal.
If $\underline{u} \cdot \underline{u} \neq 1$, then we can **normalise** the vector $\underline{u}$ by dividing by its magnitude.
To find the magnitude: $|\underline{u}| = (\underline{u} \cdot \underline{u})^{1/2}$, and the normalised vector is

$$\hat{\underline{u}} = \underline{u}/|\underline{u}|.$$

Example

$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ has magnitude $(1+1)^{1/2} = 2^{1/2}$ so the normalised vector is $\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$.

6.5.3 Orthonormal vectors

A set of vectors is **orthonormal** if they are mutually orthogonal and each has magnitude 1.

$$\text{i.e. } \underline{v}_i \cdot \underline{v}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad \text{for } i = 1, \cdots, N, \; j = 1, \cdots, N.$$

6.6 Easy to write a vector as a linear combination of orthonormal vectors

If we have a set of orthonormal vectors $\hat{\underline{v}}_1, \cdots, \hat{\underline{v}}_N$ and we are asked to find $\alpha_1, \cdots, \alpha_N$ such that $\underline{w} = \alpha_1 \hat{\underline{v}}_1 + \cdots + \alpha_N \hat{\underline{v}}_N$ then there is no need to do row reduction, etc. because

$$\begin{aligned} \underline{w} \cdot \hat{\underline{v}}_i &= \alpha_1 \hat{\underline{v}}_1 \cdot \hat{\underline{v}}_i + \cdots + \alpha_i \hat{\underline{v}}_i \cdot \hat{\underline{v}}_i + \cdots + \alpha_N \hat{\underline{v}}_N \cdot \hat{\underline{v}}_i \\ &= \alpha_i \end{aligned}$$

so we can find the coefficients easily by dot products. This is like the calculation of Fourier coefficients.

6.7 Creating orthonormal sets

It is often useful to convert a set of linearly independent vectors into an orthonormal set. We will look first at how to do this by example.

Example

$$\underline{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \ \underline{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

These are independent, but $\underline{v}_1 \cdot \underline{v}_2 \neq 0$, i.e. they are not orthogonal.

To start with we will not worry about normalising our vectors: this can be done at the end.

To create an orthogonal set $\underline{g}_1, \underline{g}_2$ that is a linear combination of $\underline{v}_1, \underline{v}_2$:

- Put $\underline{g}_1 = \underline{v}_1$. It does not matter which is first, but it can be simpler if we choose a simple $\underline{v}_1$, e.g. one with lots of zeroes. In this case neither choice is better.
- Try $\underline{g}_2 = \underline{v}_2 + \alpha \underline{v}_1$.

We want

$$\begin{aligned} \underline{g}_1 \cdot \underline{g}_2 = 0 \quad &\Longrightarrow \quad \underline{v}_1 \cdot \underline{v}_2 + \alpha \underline{v}_1 \cdot \underline{v}_1 = 0 \\ &\Longrightarrow \quad \alpha = -\frac{\underline{v}_1 \cdot \underline{v}_2}{|\underline{v}_1|^2}. \end{aligned}$$

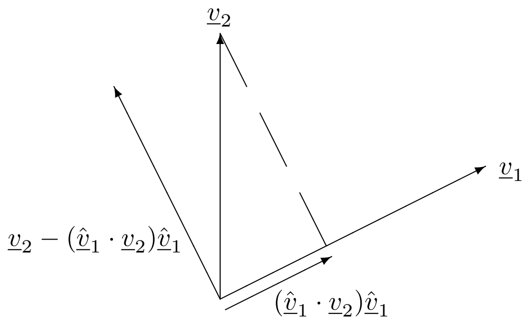
Hence

$$\begin{aligned} \underline{g}_2 &= \underline{v}_2 - \frac{\underline{v}_1 \cdot \underline{v}_2}{|\underline{v}_1|^2} \underline{v}_1 \\ &= \underline{v}_2 - \frac{\underline{v}_1 \cdot \underline{v}_2}{|\underline{v}_1|} \hat{\underline{v}}_1 \\ &= \underline{v}_2 - (\hat{\underline{v}}_1 \cdot \underline{v}_2) \hat{\underline{v}}_1. \end{aligned}$$

In our example,

$$\underline{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{5} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix}.$$

Geometrically:



6.7.1 The general process: Gram-Schmidt

Given linearly independent vectors $\underline{v}_1, \cdots, \underline{v}_N$, we will construct orthogonal $\underline{g}_1, \cdots, \underline{g}_N$ that are linear combinations of the $\underline{v}_i$.

$$
\begin{aligned}
\text{Put } \underline{g}_1 &= \underline{v}_1 \\
\text{Then } \underline{g}_2 &= \underline{v}_2 - (\underline{v}_2 \cdot \hat{\underline{g}}_1)\hat{\underline{g}}_1 \qquad = \underline{v}_2 - \frac{(\underline{v}_2 \cdot \underline{v}_1)}{(\underline{v}_1 \cdot \underline{v}_1)}\underline{v}_1 \text{ as before} \\
\underline{g}_3 &= \underline{v}_3 - (\underline{v}_3 \cdot \hat{\underline{g}}_1)\hat{\underline{g}}_1 - (\underline{v}_3 \cdot \hat{\underline{g}}_2)\hat{\underline{g}}_2 \\
&= \underline{v}_3 - \frac{(\underline{v}_3 \cdot \underline{g}_1)}{(\underline{g}_1 \cdot \underline{g}_1)}\underline{g}_1 - \frac{(\underline{v}_3 \cdot \underline{g}_2)}{(\underline{g}_2 \cdot \underline{g}_2)}\underline{g}_2 \\
\underline{g}_4 &= \underline{v}_4 - (\underline{v}_4 \cdot \hat{\underline{g}}_1)\hat{\underline{g}}_1 - (\underline{v}_4 \cdot \hat{\underline{g}}_2)\hat{\underline{g}}_2 - (\underline{v}_4 \cdot \hat{\underline{g}}_3)\hat{\underline{g}}_3
\end{aligned}
$$

and so on.
Let us check orthogonality (one example):

$$
\begin{aligned}
\underline{g}_3 \cdot \underline{g}_2 &= \underline{v}_3 \cdot \underline{g}_2 - (\underline{v}_3 \cdot \hat{\underline{g}}_1)\hat{\underline{g}}_1 \cdot \underline{g}_2 - (\underline{v}_3 \cdot \hat{\underline{g}}_2)\hat{\underline{g}}_2 \cdot \underline{g}_2 \\
&= \underline{v}_3 \cdot \underline{g}_2 - 0 - (\underline{v}_3 \cdot \hat{\underline{g}}_2)|\underline{g}_2| \\
&= 0 \;\; \text{OK.}
\end{aligned}
$$

Example

$$
\underline{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \underline{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.
$$

$$
\underline{g}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}. \quad |\underline{g}_1|^2 = 3. \quad \underline{v}_2 \cdot \underline{g}_1 = 2.
$$

$$
\underline{g}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.
$$

$$\underline{g}_2 \cdot \underline{g}_2 = \frac{6}{9} = \frac{2}{3}. \quad \underline{v}_3 \cdot \underline{g}_1 = 2. \quad \underline{v}_3 \cdot \underline{g}_2 = \frac{5}{3}.$$

$$\underline{g}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \frac{(5/3)}{(2/3)}\frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \frac{5}{6}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Note we can choose any multiple of these calculated vectors: so let us have

$$\underline{g}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \underline{g}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \underline{g}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Check orthogonality:

$$\begin{aligned} \underline{g}_1 \cdot \underline{g}_2 &= 0 \\ \underline{g}_1 \cdot \underline{g}_3 &= 0 \\ \underline{g}_2 \cdot \underline{g}_3 &= 0. \end{aligned}$$

7 Eigenvectors and eigenvalues

We already know that an eigenvalue-eigenvector pair λ, $\underline{v}$ satisfies

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{v} = \underline{0}.$$

Let's look at some more examples.

Example of repeated eigenvalues

$$\underline{\underline{A}} = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$|\underline{\underline{A}} - \lambda \underline{\underline{I}}| = \begin{vmatrix} 3-\lambda & 2 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (3-\lambda)^2(1-\lambda)$$

so if $|\underline{\underline{A}} - \lambda \underline{\underline{I}}| = 0$ then $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 3$.
The eigenvalue $\lambda = 3$ has multiplicity 2 (i.e. it appears twice).
Look for the eigenvector of $\lambda_1 = 1$:

$$(\underline{\underline{A}} - \lambda_1 \underline{\underline{I}})\underline{v}_1 = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This is easy to solve as it happens to be in echelon form: there is one solution

$$\underline{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Now the eigenvector(s) of $\lambda_2 = 3$:

$$(\underline{\underline{A}} - \lambda_2 \underline{\underline{I}})\underline{v}_2 = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Again, there is just one solution:

$$\underline{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

In this case there are only two eigenvectors; note that they are linearly independent.

In this situation, where a repeated eigenvalue has only one eigenvector $\underline{v}$, we can find further linearly independent "generalised eigenvectors" by considering, for example,

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{w} = \underline{v}, \quad \Longrightarrow (\underline{\underline{A}} - \lambda \underline{\underline{I}})^2 \underline{w} = \underline{0}.$$

In our case we have

$$\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

which gives

$$\underline{w} = \begin{pmatrix} k \\ 1/2 \\ 0 \end{pmatrix}.$$

We can pick any value of k and get the generalised eigenvector: the part we are adding on with k is always just a multiple of the standard eigenvector $\underline{v}$.

7.1 Sets of linear ODEs

Suppose we are trying to solve the coupled linear ODEs:

$$dx/dt = 3x + 5y + 2 \qquad dy/dt = 5x + 3y + 3$$

We can write this in a vector form:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + 5y + 2 \\ 5x + 3y + 3 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$

and if we set

$$\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix}$$

then the whole equation becomes

$$\frac{d\underline{v}}{dt} = \underline{\underline{A}}\,\underline{v} + \underline{b} \qquad \text{with } \underline{\underline{A}} = \begin{pmatrix} 3 & 5 \\ 5 & 3 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Revision

Recall, if we are solving a linear first-order ODE with constant coefficients, it will be of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + b, \quad \text{which we also write as} \quad \dot{x} = ax + b.$$

- We look first at the **homogeneous** equation $\dot{x} = ax$ and try solutions of the form $x = e^{\lambda t}$
- Then we look for a **particular solution** of our full equation, trying something like the "right hand side": in this case a constant.

Returning to our example system, let us try a solution of the form

$$\underline{v} = \underline{v}_0 e^{\lambda t}$$

in the homogeneous equation

$$\frac{d\underline{v}}{dt} = \underline{\underline{A}}\,\underline{v}.$$

This gives

$$\dot{\underline{v}} = \lambda \underline{v}_0 e^{\lambda t} \implies \underline{\underline{A}}\,\underline{v}_0 e^{\lambda t} = \lambda \underline{v}_0 e^{\lambda t},$$

in other words, $\underline{v}_0$ must be an eigenvector of $\underline{\underline{A}}$ with eigenvalue λ.

For this example, $\begin{vmatrix} 3-\lambda & 5 \\ 5 & 3-\lambda \end{vmatrix} = (3-\lambda)(3-\lambda) - 25 = \lambda^2 - 6\lambda - 16 = (\lambda - 8)(\lambda + 2)$ so our matrix has eigenvalues $\lambda_1 = 8$, $\lambda_2 = -2$.

$$\lambda_1 = 8 \implies \begin{pmatrix} -5 & 5 \\ 5 & -5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \underline{0} \implies \underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -2 \implies \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \underline{0} \implies \underline{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Then the general solution of the homogeneous equation is

$$\underline{v} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{8t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$$

where we have **two** unknown constants because there were originally two first-order equations, which is loosely equivalent to one second-order equation.

Constant terms in the governing equation

We now have the general solution to the homogeneous equation: but we're not trying to solve the homogeneous equation, we want to solve the full equation

$$\frac{d\underline{v}}{dt} = \underline{\underline{A}}\,\underline{v} + \underline{b}.$$

Just as for ordinary ODEs, we try something that looks like the extra function on the right: in this case $\underline{b}$ is a constant vector so we try a constant vector.

For our example, we want a solution to

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

so we try a constant vector

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad \frac{d}{dt}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 3 & 5 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

which can then be written as an augmented matrix for α and β:

$$\left(\begin{array}{cc|c} 3 & 5 & -2 \\ 5 & 3 & -3 \end{array}\right) \qquad \begin{array}{l} R1 \to R1/3 \\ R2 \to R2 - R1/3 \end{array} \qquad \left(\begin{array}{cc|c} 1 & 5/3 & -2/3 \\ 0 & -16/3 & 1/3 \end{array}\right)$$

The bottom row gives $-16\beta/3 = 1/3$ so $\beta = -1/16$; then the top row gives $\alpha + 5\beta/3 = -2/3$, $\alpha - 5/48 = -2/3$, $\alpha = -9/16$. The particular solution is

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -9/16 \\ -1/16 \end{pmatrix}$$

and the general solution to the governing equation is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{8t} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-2t} + \begin{pmatrix} -9/16 \\ -1/16 \end{pmatrix}.$$

7.1.1 What if there is a repeated eigenvalue?

Beware: sometimes, if there is a repeated eigenvalue, there may not be N eigenvectors to the N-equation system.

Example:

$$\begin{array}{rcrcr} \dot{x} & = & 3x & + & y \\ \dot{y} & = & & & 3y \end{array} \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

When we look for the eigenvalues of the matrix, we get $\lambda = 3$ twice. When we look for the eigenvectors the only solution we find is

$$\underline{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

This gives us a single solution

$$\underline{v} = c_1\underline{v}_1 e^{3t} \qquad (\text{or } x = c_1 e^{3t},\ y = 0).$$

To find another solution, we try the form

$$\begin{aligned} \underline{v} &= (t\underline{v}_1 + \underline{w})e^{\lambda t} \\ \underline{\dot{v}} &= \underline{v}_1 e^{\lambda t} + \lambda(t\underline{v}_1 + \underline{w})e^{\lambda t} = (t\lambda\underline{v}_1 + \underline{v}_1 + \lambda\underline{w})e^{\lambda t} \\ \underline{\underline{A}}\,\underline{v} &= \underline{\underline{A}}(t\underline{v}_1 + \underline{w})e^{\lambda t} = (t\underline{\underline{A}}\,\underline{v}_1 + \underline{\underline{A}}\,\underline{w})e^{\lambda t} = (t\lambda\underline{v}_1 + \underline{\underline{A}}\,\underline{w})e^{\lambda t} \end{aligned}$$

so this form satisfies $\dot{\underline{v}} = \underline{\underline{A}}\,\underline{v}$ if

$$\underline{\underline{A}}\,\underline{w} = \underline{v}_1 + \lambda\underline{w} \implies (\underline{\underline{A}} - \lambda\underline{\underline{I}})\underline{w} = \underline{v}_1$$

so $\underline{w}$ is the **generalised eigenvector** associated with λ.

For our example, this means

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies \underline{w} = \begin{pmatrix} \alpha \\ 1 \end{pmatrix} = \alpha\underline{v}_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Any choice of α will do; for convenience we choose $\alpha = 0$ so that $\underline{w}$ is simple and $\underline{v}_1 \cdot \underline{w} = 0$.

Now we have two independent solutions to combine for the general solution:

$$\underline{v} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t} + c_2 \left[t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] e^{3t},$$

where c_1 and c_2 are scalar constants.

7.1.2 Determining constants from initial conditions

We have found the general solution to several problems: we always had some undetermined constants at the end. Just as with ordinary ODEs, finding the constants from the initial conditions is the **last thing we do**.

Example

Suppose we had found the general solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{8t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + \begin{pmatrix} -9/16 \\ -1/16 \end{pmatrix}$$

and we were given the initial condition

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -9/16 \\ -33/16 \end{pmatrix} \qquad \text{at} \qquad t = 0.$$

In our general solution, when $t = 0$ we have

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -9/16 \\ -1/16 \end{pmatrix}$$

so we need

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -9/16 \\ -1/16 \end{pmatrix} = \begin{pmatrix} -9/16 \\ -33/16 \end{pmatrix}$$

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

We just solve a set of linear equations to find $\underline{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$.

In this case we get $\underline{c} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ so $c_1 = -1$ and $c_2 = 1$ and the solution that satisfies the ODE system (from last week) **and** the initial conditions is

$$\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{8t} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + \begin{pmatrix} -9/16 \\ -1/16 \end{pmatrix}.$$

This method of solution is particularly simple if the system eigenvectors are orthogonal. Then it is easy to solve the algebraic equations of the type $c_1\underline{v}_1 + c_2\underline{v}_2 = \underline{d}$ using dot products: $\underline{v}_1 \cdot \underline{v}_2 = 0$ so $\underline{v}_1 \cdot (c_1\underline{v}_1 + c_2\underline{v}_2) = \underline{v}_1 \cdot \underline{d}$ gives immediately $c_1\underline{v}_1 \cdot \underline{v}_1 = \underline{v}_1 \cdot \underline{d}$ and the solution

$$c_1 = \frac{\underline{v}_1 \cdot \underline{d}}{\underline{v}_1 \cdot \underline{v}_1} \quad \text{and similarly} \quad c_2 = \frac{\underline{v}_2 \cdot \underline{d}}{\underline{v}_2 \cdot \underline{v}_2}.$$

Summary

First we look at the homogeneous equation and find the general solution, using the eigenvalues, eigenvectors and generalised eigenvectors of the matrix.

Second we sort our a particular solution by trying a constant vector.

Finally we impose the initial conditions to sort out our constants.

8 Matrix Diagonalisation

If an $N \times N$ matrix $\underline{\underline{A}}$ has N linearly independent eigenvectors $\underline{v}_n$, put

$$\underline{\underline{V}} = \begin{pmatrix} \underline{v}_1 & \cdots & \underline{v}_N \end{pmatrix}.$$

Then

$$\underline{\underline{A}}\,\underline{\underline{V}} = \begin{pmatrix} \lambda_1\underline{v}_1 & \cdots & \lambda_N\underline{v}_N \end{pmatrix} = \underline{\underline{V}} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_N \end{pmatrix} = \underline{\underline{V}}\,\underline{\underline{\Lambda}}.$$

[Note that $\underline{\underline{V}}\,\underline{\underline{\Lambda}} \neq \underline{\underline{\Lambda}}\,\underline{\underline{V}}$; the order of multiplication of matrices is important.]

As the vectors $\underline{v}_n$ are linearly independent, $|\underline{\underline{V}}| \neq 0$ and we can invert $\underline{\underline{V}}$ to form $\underline{\underline{V}}^{-1}$. Then

$$\underline{\underline{V}}^{-1}\underline{\underline{A}}\,\underline{\underline{V}} = \underline{\underline{V}}^{-1}\underline{\underline{V}}\,\underline{\underline{\Lambda}} = \underline{\underline{\Lambda}}.$$

Example

$$\text{For } \underline{\underline{A}} = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix},$$

$$|\underline{\underline{A}} - \lambda\underline{\underline{I}}| = (5-\lambda)(2-\lambda) - 4 = \lambda^2 - 7\lambda + 6 = (\lambda-1)(\lambda-6)$$

so the matrix has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 6$.

$$\lambda_1 = 1 : \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \underline{0} \implies \underline{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\lambda_2 = 6 : \begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \underline{0} \implies \underline{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\text{Now } \underline{\underline{V}} = \begin{pmatrix} \underline{v}_1 & \underline{v}_2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}; \quad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}.$$

Check:

$$\underline{\underline{V}}^{-1} = \frac{1}{|\underline{\underline{V}}|} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.$$

$$\underline{\underline{A}}\,\underline{\underline{V}} = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -12 \\ 2 & 6 \end{pmatrix}.$$

$$\underline{\underline{V}}^{-1}\underline{\underline{A}}\,\underline{\underline{V}} = \frac{1}{5} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -12 \\ 2 & 6 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 5 & 0 \\ 0 & 30 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} = \underline{\underline{\Lambda}}.$$

Expression for $\underline{\underline{A}}$

Since $\underline{\underline{A}}\,\underline{\underline{V}} = \underline{\underline{V}}\,\underline{\underline{\Lambda}}$, we can multiply on the right by $\underline{\underline{V}}^{-1}$ to have

$$\underline{\underline{A}} = \underline{\underline{V}}\,\underline{\underline{\Lambda}}\,\underline{\underline{V}}^{-1}.$$

Summary of diagonalisation of a matrix

- Find eigenvectors and eigenvalues: this only works if we have N eigenvectors
- $\underline{\underline{V}} = \begin{pmatrix} \underline{v}_1 & \cdots & \underline{v}_N \end{pmatrix}$
- $\underline{\underline{\Lambda}} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_N \end{pmatrix}.$
- Calculate $\underline{\underline{V}}^{-1}$
- Check $\underline{\underline{V}}^{-1}\underline{\underline{A}}\,\underline{\underline{V}} = \underline{\underline{\Lambda}}$.

Common special case: $\underline{\underline{A}}$ symmetric

If $\underline{\underline{A}}$ is **symmetric**, that is, $\underline{\underline{A}}^\top = \underline{\underline{A}}$, then its eigenvalues are real.
Also, the eigenvectors $\underline{v}_i$ and $\underline{v}_j$ are **orthogonal** if $\lambda_i \neq \lambda_j$.
If further we **normalise** the eigenvectors, so $\underline{v}_i \cdot \underline{v}_i = 1$, then

$$\underline{v}_i \cdot \underline{v}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad \text{and} \quad \underline{v}_i^\top \underline{v}_j = \underline{v}_i \cdot \underline{v}_j.$$

Then

$$\underline{\underline{V}}^\top \underline{\underline{V}} = \begin{pmatrix} \underline{v}_1^\top \\ \vdots \\ \underline{v}_N^\top \end{pmatrix} \begin{pmatrix} \underline{v}_1 & \cdots & \underline{v}_N \end{pmatrix} = \begin{pmatrix} \underline{v}_1^\top \underline{v}_1 & \cdots & \underline{v}_1^\top \underline{v}_N \\ \vdots & \ddots & \vdots \\ \underline{v}_N^\top \underline{v}_1 & \cdots & \underline{v}_N^\top \underline{v}_N \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\text{so} \quad \boxed{\underline{\underline{V}}^\top = \underline{\underline{V}}^{-1}.}$$

Example

In the previous example

$$\underline{\underline{A}} = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}, \qquad \lambda_1 = 1, \qquad \underline{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad \lambda_2 = 6, \qquad \underline{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

We can see that $\underline{v}_1 \cdot \underline{v}_2 = 0$. These vectors both have length 5 so the normalised versions are divided by $\sqrt{5}$:

$$\underline{\underline{V}} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}, \qquad \underline{\underline{V}}^\top = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix},$$

$$\underline{\underline{V}}^\top \underline{\underline{V}} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underline{\underline{I}}.$$

8.1 Relation to ODEs

Suppose we have an ODE system in matrix form in which the matrix has a complete set of N linearly independent eigenvectors, and so can be reduced to the diagonal matrix $\underline{\underline{\Lambda}}$:

$$\dot{\underline{v}} = \underline{\underline{A}}\,\underline{v} + \underline{b} \qquad \underline{\underline{A}}\,\underline{\underline{V}} = \underline{\underline{V}}\,\underline{\underline{\Lambda}} \qquad \underline{\underline{V}} = \begin{pmatrix} \underline{v}_1 & \cdots & \underline{v}_N \end{pmatrix}.$$

Now put $\underline{v} = \underline{\underline{V}}\,\underline{X}$. This gives $\dot{\underline{v}} = \underline{\underline{V}}\,\dot{\underline{X}}$ and so

$$\underline{\underline{V}}\,\dot{\underline{X}} = \underline{\underline{A}}\,\underline{\underline{V}}\,\underline{X} + \underline{b} \implies \dot{\underline{X}} = \underline{\underline{V}}^{-1}\underline{\underline{A}}\,\underline{\underline{V}}\,\underline{X} + \underline{\underline{V}}^{-1}\underline{b} \implies \dot{\underline{X}} = \underline{\underline{\Lambda}}\,\underline{X} + \underline{\underline{V}}^{-1}\underline{b}$$

This is now an uncoupled system: e.g. in two dimensions

$$\underline{\underline{V}} = \begin{pmatrix} \underline{v}_1 & \underline{v}_2 \end{pmatrix} \qquad \underline{\underline{\Lambda}} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \underline{v} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \underline{X} = \begin{pmatrix} X \\ Y \end{pmatrix} \qquad \underline{\underline{V}}^{-1}\underline{b} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

(it is particularly easy to find α and β if $\underline{\underline{A}}$ is symmetric and $\hat{\underline{\underline{V}}}^{-1} = \hat{\underline{\underline{V}}}^\top$) then

$$\begin{aligned} \dot{X} &= \lambda_1 X + \alpha \\ \dot{Y} &= \lambda_2 Y + \beta \end{aligned}$$

X does not depend on Y and *vice versa*. We can easily solve each equation independently.

Example

$$\dot{x} = 3x + 4y \qquad \underline{\underline{A}} = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \qquad \underline{\underline{\Lambda}} = \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix} \qquad \underline{\underline{V}} = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$$
$$\dot{y} = 4x - 3y$$

using the eigenvalues and eigenvectors we calculated earlier for the same example.
Then putting $\underline{v} = \underline{\underline{V}}\,\underline{X}$ means

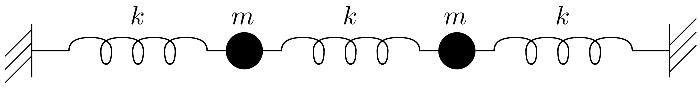
$$\begin{aligned} x &= 2X + Y \\ y &= X - 2Y \end{aligned}$$

and the diagonal system is

$$\begin{aligned} \dot{X} &= 5X \\ \dot{Y} &= -5Y, \end{aligned}$$

an uncoupled system.

8.2 A real linear system: 3 springs



The masses (both with mass m) have displacements from equilibrium x_1 and x_2. Then the differential equations governing the system are

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 + k(x_2 - x_1) \\ m\ddot{x}_2 &= -kx_2 - k(x_2 - x_1) \end{aligned}$$

Because these are second order equations, we're not quite ready to use our theory: so we introduce a new set of variables, including two new ones:

$$y_1 = x_1 \qquad y_2 = x_2 \qquad y_3 = \dot{x}_1 \qquad y_4 = \dot{x}_2.$$

It's easy to write down the two extra equations we need:

$$\dot{y}_1 = y_3 \qquad \dot{y}_2 = y_4.$$

So now we have a four-by-four system:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2k/m & k/m & 0 & 0 \\ k/m & -2k/m & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

If we put $a^2 = k/m$ then the system is

$$\dot{\underline{y}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2a^2 & a^2 & 0 & 0 \\ a^2 & -2a^2 & 0 & 0 \end{pmatrix} \underline{y} = \underline{\underline{A}}\,\underline{y}.$$

The determinant is $|\underline{\underline{A}}| = 3a^4$, and to find the eigenvalues:

$$|(\underline{\underline{A}} - \lambda \underline{\underline{I}})| = \lambda^4 + 4\lambda^2 a^2 + 3a^4 = (\lambda^2 + a^2)(\lambda^2 + 3a^2).$$

The roots of this equation are

$$\lambda_1 = ia, \qquad \lambda_2 = -ia, \qquad \lambda_3 = i\sqrt{3}a, \qquad \lambda_4 = -i\sqrt{3}a,$$

and the corresponding eigenvectors

$$\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \\ ia \\ ia \end{pmatrix}, \qquad \underline{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -ia \\ -ia \end{pmatrix}, \qquad \underline{v}_3 = \begin{pmatrix} 1 \\ -1 \\ \sqrt{3}ia \\ -\sqrt{3}ia \end{pmatrix}, \qquad \underline{v}_4 = \begin{pmatrix} 1 \\ -1 \\ -\sqrt{3}ia \\ \sqrt{3}ia \end{pmatrix}.$$

The general solution to the problem is then

$$\begin{pmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \\ ia \\ ia \end{pmatrix} e^{iat} + c_2 \begin{pmatrix} 1 \\ 1 \\ -ia \\ -ia \end{pmatrix} e^{-iat} + c_3 \begin{pmatrix} 1 \\ -1 \\ \sqrt{3}ia \\ -\sqrt{3}ia \end{pmatrix} e^{i\sqrt{3}at} + c_4 \begin{pmatrix} 1 \\ -1 \\ -\sqrt{3}ia \\ \sqrt{3}ia \end{pmatrix} e^{-i\sqrt{3}at}.$$

We are really only interested in our original variables x_1 and x_2, for which

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{iat} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-iat} + c_3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\sqrt{3}at} + c_4 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i\sqrt{3}at}.$$

There are four modes here: but they come in two complex conjugate pairs.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left[c_1 e^{iat} + c_2 e^{-iat}\right] + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \left[c_3 e^{i\sqrt{3}at} + c_4 e^{-i\sqrt{3}at}\right].$$

There are two 'normal' modes of oscillation going on:

- A mode in which $x_1 = x_2$: the middle spring is not stretched and the particles oscillate with frequency a
- A mode in which $x_1 = -x_2$: the middle of spring 2 does not move and the particles oscillate with frequency $\sqrt{3}a$.

We can write down the general solution in **real** form:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} (d_1 \cos\omega_1 t + d_2 \sin\omega_1 t) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} (d_3 \cos\omega_2 t + d_4 \sin\omega_2 t)$$

in which $\omega_1^2 = a^2$, $\omega_2^2 = 3a^2$ and the constants d_1, d_2, d_3 and d_4 will come from the initial conditions.

8.3 A cleverer way

With the original governing equations of the last section:

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 + k(x_2 - x_1) \\ m\ddot{x}_2 &= -kx_2 - k(x_2 - x_1) \end{aligned}$$

we can write a matrix–vector system:

$$\ddot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$$

with

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \underline{\underline{A}} = \begin{pmatrix} -2k/m & k/m \\ k/m & -2k/m \end{pmatrix} = \begin{pmatrix} -2a^2 & a^2 \\ a^2 & -2a^2 \end{pmatrix}$$

Now if we find the eigenvalues and eigenvectors of this new $\underline{\underline{A}}$ we can diagonalise it. For the eigenvalues:

$$|\underline{\underline{A}} - \lambda \underline{\underline{I}}| = \begin{vmatrix} -2a^2 - \lambda & a^2 \\ a^2 & -2a^2 - \lambda \end{vmatrix} = (-2a^2-\lambda)^2 - a^4 = (-2a^2-\lambda-a^2)(-2a^2-\lambda+a^2) = (\lambda+3a^2)(\lambda+a^2)$$

so the eigenvalues are $\lambda = -3a^2$ and $\lambda = -a^2$. For the eigenvectors:

$$\lambda_1 = -3a^2: \qquad \begin{pmatrix} a^2 & a^2 \\ a^2 & a^2 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = 0 \qquad \Longrightarrow \qquad \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

$$\lambda_2 = -a^2: \qquad \begin{pmatrix} -a^2 & a^2 \\ a^2 & -a^2 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = 0 \qquad \Longrightarrow \qquad \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We form our matrix using them as columns

$$\underline{\underline{V}} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

in order to create a new vector

$$\underline{x} = \underline{\underline{V}}\,\underline{X}.$$

With the new variables we have

$$\underline{\underline{V}}\,\ddot{\underline{X}} = \underline{\underline{A}}\,\underline{\underline{V}}\,\underline{X} \qquad \ddot{\underline{X}} = \underline{\underline{V}}^{-1}\underline{\underline{A}}\,\underline{\underline{V}}\,\underline{X}.$$

Now because $\underline{\underline{V}}$ was designed to diagonalise $\underline{\underline{A}}$,

$$\underline{\underline{V}}^{-1}\underline{\underline{A}}\,\underline{\underline{V}} = \underline{\underline{\Lambda}} = \begin{pmatrix} -3a^2 & 0 \\ 0 & -a^2 \end{pmatrix}$$

We now have two separate equations:

$$\ddot{X} = -3a^2 X \qquad X = c_1 \cos\sqrt{3}at + c_2 \sin\sqrt{3}at$$

$$\ddot{Y} = -a^2 Y \qquad Y = c_3 \cos at + c_4 \sin at$$

which then gives us the general solution in terms of the original variables:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underline{\underline{V}}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} c_1 \cos\sqrt{3}at + c_2 \sin\sqrt{3}at + c_3 \cos at + c_4 \sin at \\ -c_1 \cos\sqrt{3}at - c_2 \sin\sqrt{3}at + c_3 \cos at + c_4 \sin at \end{pmatrix}$$