B3D: Mathematics
University College London
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1 Functions of Several Variables

What you need to remember

- The definitions of plane polar coordinates ($r \geq 0$, $0 \leq \theta < 2\pi$):

  $$x = r\cos\theta \qquad y = r\sin\theta$$

  cylindrical polar coordinates ($r \geq 0$, $0 \leq \theta < 2\pi$, $-\infty < z < \infty$):

  $$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

  and spherical polar coordinates ($\rho \geq 0$, $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$):

  $$x = \rho\sin\theta\cos\phi \qquad y = \rho\sin\theta\sin\phi \qquad z = \rho\cos\theta$$

  and their meanings.
- The mixed derivatives theorem: $f_{xy} = f_{yx}$
- A *critical point* of a function of two variables $f(x, y)$ is a point where $f_x = 0$ and $f_y = 0$.
- The criteria for a critical point to be degenerate; the meanings of the terms maximum, minimum and saddle points and criteria for them.

What you need to be able to do

- Calculate partial derivatives
- Change variables using the extended chain rule
- Find and classify the critical points of a function of two variables

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2 Grad, Div and Curl

What you need to remember

- $$\underline{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$
- The physical meaning of $\underline{\nabla} f$ as the direction in which f increases most quickly
- $\underline{\nabla} f$ is perpendicular to surfaces (or contours) of constant f.
- The definitions

$$\begin{aligned} div(\underline{v}) &= \underline{\nabla} \cdot \underline{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \\ \underline{curl}(\underline{v}) &= \underline{\nabla} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_1 & v_2 & v_3 \end{vmatrix} \\ \nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \end{aligned}$$

What you need to be able to do

- Find $\underline{\nabla} f$ for a function of two or three variables
- Given $\underline{\nabla} f$, find f
- Calculate the directional derivative $\underline{u} \cdot \underline{\nabla} f$
- Calculate div and curl of a vector function.
- Calculate the Laplacian $\nabla^2 f$ of a function of x, y and z
- Given a function in polar coordinates and the standard formula for ∇^2 in those coordinates, calculate $\nabla^2 f$.

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3 Operators and the Commutator

What you need to remember

- An operator $O(f)$ is *linear* if:
$$O(f+g) = O(f) + O(g)$$
and
$$O(\lambda f) = \lambda O(f)$$
for all f and g, and constant λ.
- The order of operators is important
- The commutator of two linear operators O_A and O_B is
$$[O_A, O_B] = O_A \circ O_B - O_B \circ O_A.$$

What you need to be able to do

- Identify an operator as linear or nonlinear
- Work with operators which act on any of:
  - a single variable
  - functions of a single variable
  - functions of several variables
  - constant vectors
  - vector functions
- Compose two operators together
- Calculate the commutator of a pair of linear operators

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4 Ordinary Differential Equations

What you need to remember

- For a first-order linear equation
$$\frac{\mathrm{d}f}{\mathrm{d}x} + p(x)f = q(x),$$
use the integrating factor
$$I(x) = \exp\left[\int^x p(x)\,\mathrm{d}x\right].$$
- For a constant-coefficients equation, we find the CF (from the homogeneous equation) and the PI and add them together.
- You need to learn the standard forms of PI to try for a given right-hand-side of the ODE.
- If your PI fails, multiply by x (or the independent variable) and try again.
- Sorting out the initial conditions is **always** the last thing you do.
- For an equation with coefficients which are powers of x (with the power matching the order of the derivative), the CF is of the form Ax^m and, if the right hand side is also of the form x^n, the trial function for the PI is Bx^n.

What you need to be able to do

- Classify an ODE according to whether or not it is linear, what order it is, and whether it has constant coefficients.
- Solve a first-order equation using an integrating factor.
- Solve a constant-coefficients homogeneous equation using trial functions $e^{\lambda x}$, and $xe^{\lambda x}$ if λ is a repeated root.
- Solve a constant-coefficients inhomogeneous equation by finding the CF (as above) and the PI.
- Solve a linear equation where the coefficient of the rth derivative is a multiple of x^r.

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5 Fourier series

What you need to remember

- The definition of a Fourier series for a function with period $2L$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

  and for a function with period 2π:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right).$$

- The formulae for the Fourier coefficients (period $2L$):

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x.$$

- The special cases of an odd function:

$$a_n = 0 \qquad b_n = \frac{1}{2L} \int_{0}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x.$$

  and an even function:

$$a_n = \frac{1}{2L} \int_{0}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x \qquad b_n = 0.$$

What you need to be able to do

- Given a periodic function, sketch it.
- Identify a function as even or odd.
- Calculate the Fourier coefficients by carrying out the integrals.
- Differentiate or integrate a Fourier series once you know the coefficients.
- Manipulate equations involving Fourier series (e.g. Parseval's theorem).

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6 Linear equations and vectors

What you need to remember

- The **rank** of a matrix is the number of non-zero rows when it is reduced to echelon form.
- For an augmented matrix, the number of variables minus the rank of the matrix gives the number of parameters in the solution.
- If the augmented matrix reduces to one where the last row has zeros on the left and a non-zero on the right, the system has no solution.
- Vectors are linearly independent if there is no non-trivial linear combination of them that sums to zero:
  $$\alpha_1 \underline{v}_1 + \cdots + \alpha_N \underline{v}_N = \underline{0}$$
  has $\alpha_1 = \cdots = \alpha_N = 0$ as the **only** solution.
- The opposite is linearly dependent: there is some non-trivial linear combination that sums to zero.
- A set of vectors are **orthonormal** if each has length 1 and each pair are orthogonal:
  $$\underline{v}_i \cdot \underline{v}_j = 0 \quad \text{for } i \neq j; \qquad \underline{v}_i \cdot \underline{v}_i = 1.$$

What you need to be able to do

- Form an augmented matrix from a set of linear equations and reduce it to echelon form.
- Construct the solution (including parameters) to a linear system with a non-unique solution.
- Write a vector as a sum of other vectors, i.e. solve
  $$\underline{w} = \alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 + \alpha_3 \underline{v}_3.$$
  as a set of linear equations for the α_i, using an augmented matrix with columns made out of the vectors $\underline{v}_i$ and echelon form.
- Make a vector into a unit vector by dividing by its magnitude.
- Use the Gram-Schmidt process to create a set of orthonormal vectors.

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7 Eigenvectors and eigenvalues

What you need to remember

- A square matrix $\underline{\underline{A}}$ has an eigenvector $\underline{v}$ with associated eigenvalue λ if

$$\underline{\underline{A}}\,\underline{v} = \lambda \underline{v}.$$

- An eigenvalue can be zero but an eigenvector can't.
- If all the eigenvalues of a symmetric matrix are different, then you can use a matrix whose columns are eigenvectors of the matrix to diagonalise it.
- The set of linear first-order differential equations

$$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} + \underline{b}$$

  is solved by

$$\underline{x} = -\underline{\underline{A}}^{-1}\underline{b} + \underline{v}_1 \exp \lambda_1 t + \underline{v}_2 \exp \lambda_2 t + \cdots$$

  where $\underline{v}_1$, $\underline{v}_2$, etc. are the eigenvectors of $\underline{\underline{A}}$ with eigenvalues λ_1, λ_2 etc. [This only works if $\underline{\underline{A}}$ has a complete set of distinct eigenvalues; otherwise we must use generalised eigenvectors.]

What you need to be able to do

- Find the eigenvalues and eigenvectors of a square matrix (up to 3 by 3).
- Given a symmetric matrix $\underline{\underline{A}}$ with distinct eigenvalues, find a matrix $\underline{\underline{V}}$ for which

$$\underline{\underline{V}}^{\top} = \underline{\underline{V}}^{-1} \qquad \text{and} \qquad \underline{\underline{V}}^{-1}\underline{\underline{A}}\,\underline{\underline{V}} = \underline{\underline{\Lambda}}$$

  and $\underline{\underline{\Lambda}}$ is a diagonal matrix with eigenvalues of $\underline{\underline{A}}$ on the diagonal. $\underline{\underline{V}}$ will have columns consisting of unit eigenvectors of $\underline{\underline{A}}$.
- Solve a system of coupled linear first-order differential equations as given above.
