B3D

Example Sheet 3.

Handed out Monday 23 January 2006.
Due in before the lecture on Monday 30 January 2006.

1. Consider the function $f(x, y) = 2x^2 + 3xy + xy^2 - y^3$.

   (a) Find $\underline{\nabla} f$.

   (b) At the point P where $x = 0$ and $y = 1$, find the rate of increase of f in the direction of the vector $\underline{v} = 3\underline{i} + 4\underline{j}$.

   (c) If f represents the height of a landscape and you are standing on the ground above P and wish to walk uphill as steeply as possible, which way should you walk? Give your answer as a unit vector.

2. Find $\underline{\nabla} f$ for the following functions:

   (a) $f(x, y) = 3x^2 + 4y^2$

   (b) $f(x, y, z) = xyz + e^y + \sin xy$

   (c) $f(x, y, z) = x^2 + y^2 + z^2$.

3. For this question, you may like to use the fact that in three dimensions, the vector $\underline{\nabla} f$ is perpendicular to any surface $f =$ constant.

   (a) Find a vector field which is perpendicular to the surface
   $$x^3 + y^2 + z = 4.$$

   (b) Find a vector field which is perpendicular to the surface
   $$z = x^2 - y^2.$$

4. For the vector function
   $$\underline{v}(x, y, z) = (x^2 + y^2, 2xy, xyz)$$

   (a) Calculate the divergence of $\underline{v}$: that is, find $\underline{\nabla} \cdot \underline{v}$.

   (b) Calculate the curl of $\underline{v}$: i.e. find $\underline{\nabla} \times \underline{v}$.

5. Calculate the Laplacian of the following functions, using the formulae given below if the function is expressed in polar coordinates:

   (a) $f_1(x, y, z) = 3x^2 + y^2 + \sin z$

   (b) $f_2(r, \theta) = r \cos\theta$

   (c) $f_3(r, \theta, z) = z^2 - r^2$

   (d) $f_4(\rho, \theta, \phi) = \rho^2 \sin^2\theta \sin^2\phi$.

   The formulae for the Laplacian in polar coordinates are:

   In plane polars: $$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

   In cylindrical polars: $$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

   In spherical polars: $$\nabla^2 f = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$