B3D

Example Sheet 4.

Handed out Monday 30 January 2006.
Due in before the lecture on Monday 6 February 2006.

1. Are the following operators linear or nonlinear?

$$
\begin{array}{lcl}
O_a & : & x \to |x| \\
O_c & : & x \to x \\
O_e & : & f(x,y) \to f_x + f_y \\
O_g & : & \underline{v} \to \underline{\underline{A}}\,\underline{v}
\end{array}
\qquad
\begin{array}{lcl}
O_b & : & x \to e^x \\
O_d & : & f(x) \to \mathrm{d}^2 f/\mathrm{d}x^2 \\
O_f & : & f(x,y) \to f_x f_y \\
O_h & : & \underline{\underline{A}} \to \underline{\underline{A}}^{-1}
\end{array}
$$

2. Find expressions for the following commutator operators:
   - Operating on $f(x)$:
     (a) $[\mathrm{d}^2/\mathrm{d}x^2, x]$
     (b) $[\mathrm{d}^3/\mathrm{d}x^3, x^2]$
   - Operating on $f(x,y)$:
     (c) $[\partial/\partial x + \partial/\partial y, xy]$
     (d) $[x\partial/\partial y - y\partial/\partial x, xy]$

3. If the matrices $\underline{\underline{A}}$, $\underline{\underline{B}}$, $\underline{\underline{C}}$ and $\underline{\underline{D}}$ are defined as:

$$
\underline{\underline{A}} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \qquad
\underline{\underline{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \qquad
\underline{\underline{C}} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \qquad
\underline{\underline{D}} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}
$$

   and the operators O_A, O_B, O_C and O_D are defined as

$$
\begin{array}{lcl}
O_A & : & \underline{v} \to \underline{\underline{A}}\,\underline{v} \\
O_B & : & \underline{v} \to \underline{\underline{B}}\,\underline{v}
\end{array}
\qquad
\begin{array}{lcl}
O_C & : & \underline{v} \to \underline{\underline{C}}\,\underline{v} \\
O_D & : & \underline{v} \to \underline{\underline{D}}\,\underline{v}
\end{array}
$$

   calculate the commutators

   (a) $[O_A, O_B]$,
   (b) $[O_A, O_C]$,
   (c) $[O_A, O_D]$,
   (d) $[O_B, O_C]$,
   (e) $[O_B, O_D]$,
   (f) $[O_C, O_D]$.

4. [Exam question, 1999] In this question you need to know about the **identity function**, which leaves unchanged the object on which it operates:

$$
Id \quad : \quad f(x) \to f(x)
$$

   You also need the notation $x_1 = x$, $x_2 = y$, $x_3 = z$ for part (c).

   (a) Explain what is meant by the commutator $[L_1, L_2]$ of two linear operators L_1, L_2.

   (b) Show that, for functions of a single variable x,

$$
\left[\frac{\mathrm{d}}{\mathrm{d}x}, x\right] = Id
$$

   and find an expression for $\left[\frac{\mathrm{d}^3}{\mathrm{d}x^3}, x^3\right]$ in terms of Id, $\frac{\mathrm{d}}{\mathrm{d}x}$, $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$.

   (c) Let $\{e_1, e_2, e_3\}$ denote the standard basis of $\mathbb{R}^3$. The angular momentum operator in the e_i, e_j plane is

$$
L_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}.
$$

   Calculate $[L_{12}, L_{13}]$.