B3D

Example Sheet 9.

Handed out Tuesday 14 March 2006.
Due in before the lecture on Monday 20 March 2006.

1.

$$\underline{a} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix} \quad \underline{c} = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} \quad \underline{d} = \begin{pmatrix} 6 \\ 14 \\ 5 \end{pmatrix}.$$

Show that the vectors $\underline{a}$, $\underline{b}$, $\underline{c}$ are linearly independent.

Write $\underline{d}$ as a linear combination of $\underline{a}$, $\underline{b}$, $\underline{c}$.

2. You are given four vectors

$$\underline{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \underline{x}_2 = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \qquad \underline{x}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 3 \end{pmatrix} \qquad \text{and} \qquad \underline{w} = \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Verify that the vectors $\underline{x}_1$, $\underline{x}_2$ and $\underline{x}_3$ are linearly independent. Express the vector $\underline{w}$ as a linear combination of $\underline{x}_1$, $\underline{x}_2$ and $\underline{x}_3$.

Use the Gram-Schmidt process to create a set of three mutually orthogonal vectors $\underline{g}_1$, $\underline{g}_2$, $\underline{g}_3$ each of which is a linear combination of $\underline{x}_1$, $\underline{x}_2$ and $\underline{x}_3$. Use dot products to calculate the coefficients α_i such that

$$\underline{w} = \alpha_1 \underline{g}_1 + \alpha_2 \underline{g}_2 + \alpha_3 \underline{g}_3.$$

3. Find the eigenvalues and eigenvectors of the matrix

$$\underline{\underline{B}} = \begin{pmatrix} 2 & 6 \\ 6 & -7 \end{pmatrix}.$$

Give a matrix $\underline{\underline{V}}$ for which $\underline{\underline{B}} = \underline{\underline{V}}\,\underline{\underline{\Lambda}}\,\underline{\underline{V}}^{-1}$ for $\underline{\underline{\Lambda}}$ a diagonal matrix.

Find the inverse matrix $\underline{\underline{V}}^{-1}$ and calculate directly the two products

- $\underline{\underline{V}}^{-1}\underline{\underline{B}}\,\underline{\underline{V}}$, which should be the diagonal matrix $\underline{\underline{\Lambda}}$, and
- $\underline{\underline{V}}\,\underline{\underline{\Lambda}}\,\underline{\underline{V}}^{-1}$, verifying that you regain the original matrix $\underline{\underline{B}}$.

4. Consider the coupled ordinary differential equations

$$\begin{aligned} \dot{x} &= 2x + 6y + 2 \\ \dot{y} &= 6x - 7y - 4. \end{aligned}$$

Write these two equations in the form $\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} + \underline{b}$, explaining what $\underline{x}$, $\underline{\underline{A}}$ and $\underline{b}$ are.

Find the general solution to this matrix differential equation (using results from question 3 where possible).