Exercises

The first 12 are from the lecture notes; the answers (but not full solutions) will be available on the website `http://www.ucl.ac.uk/~ucahhwi`. You may wish to save the others for revision next term.

1. Find the first three terms of an expansion for each root of the following equation:
$$x^3 - (2-\varepsilon)x^2 - x + 2 + \varepsilon = 0.$$

2. Try a regular perturbation expansion in the following differential equation:
$$y'' + 2\varepsilon y' + (1+\varepsilon^2)y = 1, \qquad y(0) = 0, \qquad y(\pi/2) = 0.$$
Calculate the first three terms, that is, up to order ε^2. Apply the boundary conditions at each order.

3. For the equation
$$\varepsilon x^2 + x - 1 = 0,$$
calculate the root near $x = 1$, up to and including terms of order ε^3.

4. Find the distinguished scalings for the following equation:
$$\varepsilon^3 x^3 + x^2 + 2x + \varepsilon = 0.$$
and find the first two nonzero terms in the expansion of each root.

5. Find the distinguished scalings for the following equation:
$$\varepsilon x^3 + x^2 + (2-\varepsilon)x + 1 = 0.$$
and find the first two terms in the expansion of each root. [Hint: you may find an exact root – then there will be no more terms in the expansion.]

6. Find the first two terms of all four roots of
$$\varepsilon x^4 - x^2 - x + 2 = 0.$$

7. Calculate the first two terms of the solution to
$$\varepsilon\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \frac{\mathrm{d}f}{\mathrm{d}x} - f = 0$$
where derivatives are order 1 (i.e. $\delta = 1$).

8. Find the distinguished stretches for the following differential equation:
$$\varepsilon^3\frac{\mathrm{d}^3 f}{\mathrm{d}x^3} + \varepsilon\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \frac{\mathrm{d}f}{\mathrm{d}x} + f = 0$$
Find the leading-order term of each solution.

9. Consider the equation
$$\varepsilon \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + f\frac{\mathrm{d}f}{\mathrm{d}x} - f = 0$$
Find the scalings $f = \varepsilon^\alpha F$ and stretches $x = a + \varepsilon^\beta z$ at which two dominant terms balance, and sketch these balance scalings in the α–β plane. Hence determine the critical values of α and β for which all three terms balance. Give also the possible values of β if we are constrained by the boundary conditions to have $\alpha = 0$.

10. Look at the problem
$$\varepsilon \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \frac{\mathrm{d}f}{\mathrm{d}x} = \cos x$$
with boundary conditions $f(0) = 0$, $f(\pi) = 1$. Find the two distinguished stretches for this equation. Calculate the first three terms of the regular expansion, and apply the boundary condition at π to determine the constants.

    Now apply your stretch near $x = 0$. Find the first three terms of the inner solution, and apply the boundary condition at $x = 0$ to determine some of the constants in this expansion.

    Finally use an intermediate variable to match your two expressions and determine the remaining constants.

11. Show that the solution of the ODE
$$2\frac{\mathrm{d}F}{\mathrm{d}z} = 4k^2 - F^2$$
with $|F| > 2k$ is
$$F = 2k \coth\left[kz + B\right].$$

12. Consider the **advection-diffusion** equation for f (weak diffusion):
$$\underline{\nabla} \cdot [f\underline{V}] - \varepsilon\nabla^2 f = 0.$$
We will impose boundary conditions
$$\begin{aligned} fV_y - \varepsilon\frac{\partial f}{\partial y} &= 0 \quad \text{at} \quad y = 1, \\ f &= 2 \quad \text{at} \quad y = 2 \end{aligned}$$
The boundary condition at $y = 1$ corresponds to a condition of **no flux** of f through the boundary $y = 1$.

    Now suppose that the imposed velocity field is given by
$$V_x = \kappa x/y \qquad V_y = -\kappa.$$

    (a) Substitute the velocity field into the governing equation and boundary conditions.

    (b) Setting $\varepsilon = 0$, find a solution f_0 which matches the upper boundary condition at $y = 2$. [Hint: try $f_0(x, y) = g(y)$.]

(c) Substitute your solution back into the full governing equation. What can you say about the corrections to f_0 for $\varepsilon \neq 0$?

(d) The inner boundary condition at $y = 1$ is not satisfied. Assume that there is a boundary layer close to $y = 1$. How does the size of this layer scale with ε? Assume that derivatives with respect to x remain order 1.

(e) Introduce a scaled variable z to replace y near $y = 1$. Replace y in the governing equation and give your new PDE to two orders of magnitude. Do the same for the inner boundary condition at $y = 1$.

(f) Using your new PDE and boundary condition **alone**, calculate the first term of a pertubration expansion for $F(x, z) = f(x, y)$, valid within the boundary layer near $y = 1$. You will not be able to determine all the constants (or even all the functions of x) at this stage.

[Hint: if you have a PDE in which all the derivatives are with respect to z, you can solve it like an ODE in z but all the "constants of integration" must be functions of x.]

(g) Can your solution be made to match onto the outer solution as $z \to \infty$ and $y \to 1$?

(h) Continue to the next order correction in your inner expansion. What PDE must be satisfied by the next term? What boundary conditions will be applied to it?

(i) Solve your PDE for the second term in the expansion of the inner solution. Apply the boundary condition.

(j) Carry out matching between your outer and inner solutions. Thus find an ODE in x for the unknown function from the calculation of (f).

(k) Solve the ODE to complete the calculation of the leading-order term in the inner expansion. You may assume that the function is well-behaved at $x = 0$.

13. [Hinch 5.4 part 1.] Find the rescaling of x near $x = 0$ for

$$\varepsilon x^m y' + y = 1$$

in $0 < x < 1$ with $y(0) = 0$ and $0 < m < 1$.

14. [Hinch 5.4 part 2. Much harder: very optional.] Find the rescaling of x near $x = 0$ for

$$\varepsilon x y' + y = 1$$

in $0 < x < 1$ with $y(0) = 0$. Hint: instead of powers of ε, try $x = \delta(\varepsilon) z$ for a general function δ.

15. [Hinch 5.11; Van Dyke.] Calculate three terms of the outer solution of

$$(1 + \varepsilon) x^2 y' = \varepsilon((1 - \varepsilon) x y^2 - (1 + \varepsilon) x + y^3 + 2\varepsilon y^2)$$

in $0 < x < 1$ with $y(1) = 1$. Locate the non-uniformity of the asymptoticness, and hence the rescaling for an inner region. Find two terms for this inner solution.

16. [Hinch 5.6, modified.] Consider the equation

$$\varepsilon^2 y'''' - y'' = -1$$

in $-1 < x < 1$, with boundary conditions $y = y' = 0$ at $x = -1, 1$. Use the symmetry of the problem to deduce whether y is an even or odd function of x. Now, assuming that a boundary layer forms at each boundary, find the scaling for this boundary layer. Calculate the first two terms of the inner expansion near $x = 1$. Calculate the first two terms of the outer expansion. Match the two to determine your unknowns.

17. [Cole 2.3.1.]

$$\varepsilon y'' + (1 + x)y' + y = 0$$

over the range $0 \le x \le 1$ with $y(0) = 0$ and $y(1) = 1$. We will assume that the boundary layer in this problem occurs at $x = 0$. Calculate the first two terms of the outer expansion, applying the boundary condition at $x = 1$. Find the scaling for the boundary layer. Hence find the first two terms of the inner expansion near $x = 0$, applying the boundary condition at $x = 0$. Finally, match the two expansions to determine the unknown constants.